The centralizer of a nilpotent section

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Overview

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- Main results

2 Nilpotent sections over schemes
- Equidimensional nilpotent sections
- Levi factors and Bala-Carter data
- Component groups

Good characteristic

- For a field K, consider K-groups of the form

$$H_1 = H \times T;$$

  T a K-torus, H semisimple s.t. $p = \operatorname{char} K$ is *very good* for H.
- The reductive G is T-standard if it is separably isogenous to $C_{H_1}(S)$ for some H_1 and some torus $S \subset H_1$
- examples:
  - any form of GL_n is T-standard.
  - SL_n is T-standard $\iff n \not\equiv 0 \pmod p$.
- S'pose G is T-standard and let $\mathfrak{g} = \mathrm{Lie}(G)$.

Lemma

For $X \in \mathfrak{g}(K)$, the K-group scheme $C_G(X)$ is smooth over K.

In down-to-earth terms: the orbit of X is *separable.*

Richardson orbits

Theorem

(Richardson) For a parabolic subgroup P, there is $X \in \mathrm{Lie}\, R_uP(K)$ such that the orbit $\mathrm{Ad}(P)X$ is dense in $\mathrm{Lie}\, R_uP$.

- call any such X a *Richardson element* for P
- nilpotent $X \in \mathfrak{g}$ is *distinguished* if a max torus of $C_G(X)$ is central in G.
- A parabolic P with $\mathfrak{u} = \mathrm{Lie}(R_uP)$ is *distinguished* if $\dim P/R_u = \dim \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$.
- A Richardson element for P is dist. $\iff$ P is dist.

Cocharacters

Let $X \in \mathfrak{g}(K)$, choose a max torus $S \subset C_G(X)$, and let $L = C_G(S)$. Then $X \in \mathrm{Lie}(L)$ is dist. for L.

Theorem

(Premet, M.) There is a unique K-homomorphism $\phi : \mathbf{G}_m \to G$ s.t.

(i) *the image of ϕ lies in (L, L),*

(ii) $\mathrm{Ad}(\phi(t))X = t^2 X$ *for $t \in K_{\mathrm{sep}}^{\times}$,*

The parabolic subgp $P = P(X)$ with $\mathrm{Lie}(P) = \sum_{i \geq 0} \mathfrak{g}(\phi; i)$ doesn't depend on choice of S.

Remarks:

- $\mathfrak{g}(\phi, i) = \{Y \in \mathfrak{g} \mid \mathrm{Ad}(\phi(t))Y = t^i Y \quad t \in K_{\mathrm{sep}}^{\times}\}$.
- The *cocharacter* ϕ is said to be *associated with* X.
- Existence uses results of Kempf and of Rousseau in GIT
- P is called the *instability parabolic* for X

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Main results

Bala-Carter

Assume that K is separably closed. Consider pairs

$$(L, Q) \text{ with } \begin{cases} L\text{: Levi factor of a parab. of } G, \\ Q\text{: dist. parab. of } L. \end{cases}$$

Theorem

(Bala-Carter) Assoc'ing to (L, Q) the orbit $\mathrm{Ad}(G)X$ of a Richardson element X for Q gives a bijection

$$\{(L, Q)\}/G \xrightarrow{\sim} \{\mathrm{Ad}(G)\text{-orbits of nilpotent elts}\}.$$

Remarks:

- Call (L, Q) *the Bala-Carter datum* of a representative X for the corresponding orbit.

Formulation of Main Theorems

- Let F_1, F_2 be separably closed fields where char $F_1 = 0$ and char $F_2 = p > 0$.
- Let G_{F_i} be a T-standard reductive group over F_i for $i = 1, 2$.
- Assume G_{F_1} and G_{F_2} have the *same root data*.
- Let $X_i \in \mathfrak{g}_{F_i}$ be nilpotent with *the same Bala-Carter data*, and write $C_i = C_{G_{F_i}}(X_i)$ for $i = 1, 2$.

Theorem
The groups C_1 and C_2 have Levi factors with the same root data.

Theorem
The component groups C_1/C_1^o and C_2/C_2^o are isomorphic.

Reductive gp schemes and equidimensional sections

- $\mathcal{A}$: normal, Noetherian, local domain (e.g. a DVR), with fractions $K = k(\eta)$ and residues $k = k(s)$.
- G: T-standard reductive group scheme over $\mathcal{A}$
- Then $G_t = G \times_{\mathcal{A}} k(t)$ is T-standard $/k(t)$ $\quad \forall t \in \mathrm{Spec}(\mathcal{A})$.
- a section $X \in \mathfrak{g}(\mathcal{A})$ is *nilpotent* if $X(\eta) \in \mathfrak{g}(k(\eta))$ is nilp.

- **Lemma**
  For $X \in \mathfrak{g}(\mathcal{A})$ TFAE:
  1. $\dim C_G(X)_t$ *is constant for* $t \in \mathrm{Spec}(\mathcal{A})$
  2. $\dim C_G(X)_\eta = \dim C_G(X)_s$ $\quad$ *(η = generic, s = closed)*
  3. $C_G(X)$ *is smooth over* $\mathcal{A}$.

  Proof depends on smoothness of all $C_G(X)_t = C_{G_t}(X(t))$
- X is called *equidimensional* if it satisfies (1) of Lemma

The instability parabolic over $\mathcal{A}$

- A smooth $\mathcal{A}$-subgroup scheme $P \subset G$ is *parabolic* if P_t is a parabolic subgroup of G_t $\forall t \in \mathrm{Spec}(\mathcal{A})$.

Suppose $X \in \mathfrak{g}(\mathcal{A})$ is an equidimensional nilpotent section.

Proposition

There is a unique $\mathcal{A}$-parabolic subgroup scheme $P \subset G$ such that P_η is the instability parabolic for $X(\eta)$.

- the scheme of parab's of G is projective over $\mathcal{A}$. If $\mathcal{A}$ is a DVR, prop. is "for free". In gen'l, normality of $\mathcal{A}$ is req'd.
- to see the potential trouble, consider: $\mathcal{A} = k[x,y]_{(x,y)}$, $G = \mathrm{SL}_{2/\mathcal{A}}$. Then $(x : y) \in \mathbf{P}^1_{\mathcal{A}}(K)$ does not determine an $\mathcal{A}$-section of $\mathbf{P}^1_{\mathcal{A}}$.

Associated cocharacters over $\mathcal{A}$

$X \in \mathfrak{g}(\mathcal{A})$ equidim'l nilp; $P \subset G$ an $\mathcal{A}$-parabolic s.t. P_η is the instab. parabolic for $X(\eta)$.

Proposition

P_t is the instab parab for $X(t)$ for each $t \in \mathrm{Spec}(\mathcal{A})$.

Proposition

$\exists$ finite, local, étale extension $\mathcal{A} \subset \mathcal{B}$ and a $\mathcal{B}$-homomorphism $\phi : \mathbf{G}_m \to P_{/\mathcal{B}}$ s.t. ϕ_t is assoc to $X(t)$ $\forall t \in \mathrm{Spec}(\mathcal{B})$.

- the proofs depends on the following:
  if T is an $\mathcal{A}$-torus and $\phi_0 : (\mathbf{G}_m)_\eta \to T_\eta$ a $k(\eta)$ morphism, normality of $\mathcal{A}$ $\implies$ $\exists!$ $\mathcal{A}$-morphism $\phi : \mathbf{G}_m \to T$ s.t. $\phi_\eta = \phi_0$.

Levi factors and Bala-Carter data

$X \in \mathfrak{g}(\mathcal{A})$ equidim'l nilp. $C = C_G(X)$ centralizer subgroup scheme.

Corollary

1. *Locally in the étale topology, C has a Levi factor $L \subset C$; thus $L_{\bar{t}}$ is a Levi factor of $C_{\bar{t}}$ $\forall t \in \operatorname{Spec}(\mathcal{A})$.*
2. *The root datum of $L_{\bar{t}}$ is constant for $t \in \operatorname{Spec}(\mathcal{A})$.*
3. *Locally in the étale topology, $\exists$ Levi subgroup scheme $M \subset G$ and dist. parab. subgroup scheme $Q \subset M$ s.t. (M_t, Q_t) is the Bala-Carter datum of $X(t) \forall t \in \operatorname{Spec}(\mathcal{A})$.*

- first "Main Theorem" of a few slides back now follows once we know existence of enough equidimensional sections.

Richardson sections

- Suppose that G is split reductive over $\mathcal{A}$.
- (this is anyhow true locally in the étale topology.)
- given a Bala-Carter datum on some fiber G_t, it is realized [up to $G(k(\bar{t}))$-conjugacy] as (L_t, Q_t) for a subgroup scheme L which is a Levi factor of a parab of G, and $Q \subset L$ a dist. parab
- Assume $k = k(s)$ is infinite. Using a lift of a Richardson element in $\operatorname{Lie} R_u(Q)(k)$ one proves:

Theorem

$\exists$ equidimensional nilpotent section $X \in (\operatorname{Lie} R_u Q)(\mathcal{A})$ s.t. Bala-Carter datum of $X(t)$ is (L_t, Q_t).

- first "Main Theorem" follows.

Component groups: adjoint case

- Suppose that the T-standard $\mathcal{A}$-group scheme G is semisimple and *adjoint*, that $X \in \mathrm{Lie}(G)(\mathcal{A})$ is equidimensional nilpotent, and put $C = C_G(X)$.

Theorem (Mizuno, Alekseevskiĭ, Premet, M-Sommers)

The groups $C_{\bar{t}}/C^o_{\bar{t}}$ are all isomorphic, for $t \in \mathrm{Spec}(\mathcal{A})$.

- theorem implies that the sheaf C/C^o on (the étale topology of) $\mathrm{Spec}(\mathcal{A})$ is repr. by finite étale $\mathcal{A}$-group scheme.

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Component groups: extension to the general case

To prove the second “Main Theorem” for any T-standard group, need to consider *for example* the following:

- Let $f : G \to G_1$ be an étale central isogeny of $\mathcal{A}$-group schemes,
- let $X_1 = df(X)$ for $X \in \mathrm{Lie}(G)(\mathcal{A})$, $X_1 \in \mathrm{Lie}(G_1)(\mathcal{A})$.
- X is nilp., resp. equidim'l $\iff$ X_1 is nilp., resp. equidim'l
- in equidim'l case, locally in the étale topology, there are Levi factors L of $C_G(X)$ and L_1 of $C_{G_1}(X)$ such that f restricts to étale central isogeny $L \to L_1$
- Conclude: $C/C^o \simeq L/L^o$ is constant on $\mathrm{Spec}(\mathcal{A})$ iff $C_1/C_1^o \simeq L_1/L_1^o$ is constant.

Examples

- e.g. $\mathcal{A} = \mathbf{Z}_{(p)}, p \neq 2$
  - $Q_\eta = (-1,-1)$ "usual" Quaternion $K(\eta) = \mathbf{Q}$-algebra; basis $1, i, j, k$
  - $Q \subset Q_\eta$ the $\mathcal{A}$-subalg with $\mathcal{A}$-basis $1, i, j, k$ (Azumaya $\mathcal{A}$-alg)
  - $Q_s \simeq \mathrm{Mat}_2(\mathbf{F}_p)$; Q_η is division
- $G = \mathrm{GL}_{2,Q}$, an $\mathcal{A}$-form of GL_4.
- $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{2,Q}(\mathcal{A})$ is equidimensional, nilp.
- the partition of $X(t)$ is $(2,2)$ for $t = s, \eta$.
- $C_G(X)$ has $\mathcal{A}$-Levi factor L with $\mathcal{A}$-points $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in Q^\times \right\}$. Then $L \simeq \mathrm{GL}_{1,Q}$
- $L_s = \mathrm{GL}_{2/\mathbf{F}_p}$ while L_η is a non-split $\mathbf{Q}$-form of GL_2.