 3/28/2008

#8, p322, Chapter-3 review

$$A=\begin{bmatrix}-2 & 2 & -2\\ -4 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} \quad \vec{E}(t)=\begin{bmatrix}2\\ e^{2t}\\ e^{2t}\end{bmatrix} \quad \vec{x}(0)=\begin{bmatrix}3\\1\\1\end{bmatrix}$$

Solve $D\vec{x}=A\vec{x}+\vec{E}(t)$ with initial conditions.

$$p(\lambda)=\begin{vmatrix}-2-\lambda & 2 & -2\\ -4 & 2-\lambda & 0\\ 0 & 0 & 2-\lambda\end{vmatrix} \begin{aligned}&=-(\lambda-2)\left[(\lambda+2)(\lambda-2)+8\right]\\ &=-(\lambda-2)(\lambda^2+4)\end{aligned}$$

$\lambda = 2, \pm 2i$ eigenvalues

$$\underline{\lambda=2}\ \left[\begin{array}{ccc|c}-4 & 2 & -2 & 0\\ -4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \quad \vec{v}=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\underline{\lambda=-2i}\ \left[\begin{array}{ccc|c}-2+2i & 2 & -2 & 0\\ -4 & 2+2i & 0 & 0\\ 0 & 0 & 2+2i & 0\end{array}\right] \xrightarrow[2\times②]{(-2-2i)\times①} \left[\begin{array}{ccc|c}8 & -4-4i & 4+4i & 0\\ -8 & 4+4i & 0 & 0\\ 0 & 0 & 2+2i & 0\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c}1 & -\frac{1}{2}-\frac{1}{2}i & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \quad \vec{v}=\begin{bmatrix}1+i\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}1+i\\2\\0\end{bmatrix}e^{-2it}=\begin{bmatrix}1+i\\2\\0\end{bmatrix}(\cos 2t - i\sin 2t)$$

$$=\begin{bmatrix}\cos 2t - i\sin 2t + i\cos 2t + \sin 2t\\ 2\cos 2t - 2i\sin 2t\\ 0\end{bmatrix}=\begin{bmatrix}\cos 2t+\sin 2t\\ 2\cos 2t\\ 0\end{bmatrix}+i\begin{bmatrix}\cos 2t - \sin 2t\\ -2\sin 2t\\ 0\end{bmatrix}$$

$$\vec{h}_1(t)=\begin{bmatrix}0\\1\\1\end{bmatrix}e^{2t} \quad \vec{h}_2(t)=\begin{bmatrix}\cos 2t+\sin 2t\\ 2\cos 2t\\ 0\end{bmatrix} \quad \vec{h}_3(t)=\begin{bmatrix}\cos 2t-\sin 2t\\ -2\sin 2t\\ 0\end{bmatrix}.$$

Solve $\begin{bmatrix} 0 & \cos 2t+\sin 2t & \cos 2t-\sin 2t \\ e^{2t} & 2\cos 2t & -2\sin 2t \\ e^{2t} & 0 & 0 \end{bmatrix} \vec{c}\,'(t) = \begin{bmatrix} 2 \\ e^{2t} \\ e^{2t} \end{bmatrix}$

The 3rd equation is $c_1'(t)e^{2t} = e^{2t}$ or $c_1'(t) = 1$.
$\therefore$ the system simplifies to

$$\begin{bmatrix} \cos 2t+\sin 2t & \cos 2t-\sin 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix} \begin{bmatrix} c_2'(t) \\ c_3'(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_3'(t) = \frac{\cos 2t}{\sin 2t} c_2'(t)$$

$$\Rightarrow \left(\cos 2t+\sin 2t + \frac{\cos^2 2t - \cos 2t \sin 2t}{\sin 2t}\right) c_2'(t) = 2$$

$$\Rightarrow c_2'(t) = 2\sin 2t \text{ and } c_3'(t) = 2\cos 2t$$

$$c_1(t) = t, \quad c_2(t) = -\cos 2t, \quad c_3(t) = \sin 2t$$

$$\vec{p}(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t e^{2t} - \begin{bmatrix} \cos^2 2t + \cos 2t \sin 2t \\ 2\cos^2 2t \\ 0 \end{bmatrix} + \begin{bmatrix} \cos 2t \sin 2t - \sin^2 2t \\ -2\sin^2 2t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t e^{2t} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

General solution

(a) $$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \cos 2t+\sin 2t \\ 2\cos 2t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \cos 2t-\sin 2t \\ -2\sin 2t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t e^{2t} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}$$

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$$\vec{x}(0) = c_1\begin{bmatrix}0\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0&1&1\\1&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix} + \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}4\\3\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}0&1&1&4\\1&2&0&3\\1&0&0&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&0&3\\0&1&1&4\\0&-2&0&-2\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&1\\0&1&1&4\\0&0&2&6\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&3\end{array}\right] \qquad \begin{array}{l}c_1 = 1\\c_2 = 1\\c_3 = 3\end{array}$$

$$\vec{x}(t) = \begin{bmatrix}0\\1\\1\end{bmatrix}e^{2t} + \begin{bmatrix}\cos 2t + \sin 2t\\2\cos 2t\\0\end{bmatrix} + 3\begin{bmatrix}\cos 2t - \sin 2t\\-2\sin 2t\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}te^{2t} - \begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix}e^{2t} + \begin{bmatrix}4\cos 2t - 2\sin 2t\\2\cos 2t - 6\sin 2t\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}te^{2t} - \begin{bmatrix}1\\2\\0\end{bmatrix}$$

(b)

$$\vec{x}(t) = \begin{bmatrix}0\\t+1\\t+1\end{bmatrix}e^{2t} + 2\begin{bmatrix}2\cos 2t - \sin 2t\\\cos 2t - 3\sin 2t\\0\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix}$$