Exam I **Fall 2006**

No calculators, notes, or books are allowed. All answers should be in terms of real numbers and functions. Show all your work. Cross out work you do not want graded. *Please sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam.*

1. (15 points) Find the general solution of $3x' + 2x = 1$.

Solution: Solve the associated homogeneous differential equation $3x' + 2x = 0$ by separation of variables to get $x(t) = ke^{-2t/3}$. Variation of parameters means trying $x(t) = k(t)e^{-2t/3}$; this gives $k'(t) = e^{2t/3}/3$ hence $k(t) = e^{2t/3}/2 + C$ and $x(t) = 1/2 + Ce^{-2t/3}$. Alternative methods: Separation of variables directly, or integrating factors.

2. (15 points) Consider the differential equation $x\dfrac{dx}{dt} = -t^2$.

a. Find the largest rectangular region in the t, x-plane that contains the point $(0, -5)$ and on which the hypotheses of the existence and uniqueness theorem hold.

Solution: $x < 0$.

b. Is this differential equation linear? Explain!

Solution: No, because of the xx'-term—the differential equation can not be brought into the form $a_1(t)x' + a_0(t)x = E(t)$.

c. Find the solution for which $x(0) = -5$.

Solution: Separation of variables: $x^2/2 = \int x\,dx = -\int t^2\,dt = -t^3/3 + C$. Insert $t = 0$, $x = -5$ to get $25 = (-5)^2 = C$, so $x = -\sqrt{25 - 2t^3/3}$.

3. (15 points) Consider the first-order differential equation

(*) $$\frac{dx}{dt} = x^3(x+2)(x-1)^2.$$

a. Find all equilibria and draw the phase portrait of (*).

Solution: The equilibria are $-2, 0, 1$; the phase portrait is $\rightarrow - \bullet - \leftarrow \bullet \rightarrow - \bullet \rightarrow -$.

b. Which equilibria are attractors, which are repellers, and which are neither? Which are stable and which are unstable?

Solution: -2 is an attractor, 0 is a repeller, 1 is neither; -2 is stable, 0 and 1 are unstable.

c. Find a value x_0 such that the solution $x(t)$ of (*) with initial condition $x(0) = x_0$ approaches ∞ as $t \to \infty$.

Solution: 17. (Or any other number greater than 1.)

4. (5 points) Find all solutions of $(tD^2 - D)x = 0$ that are of the form t^α.

Solution: Plug in t^α to get $(\alpha(\alpha-1) - \alpha)t^{\alpha-1} = 0$ for all t, hence $\alpha = 0$ and $\alpha = 2$ are the only solutions, so 1 and t are (all) solutions of the differential equation of the form t^α.

5. (10 points) Consider the linear differential equation

(**) $$t^2x'' - 6tx' + 12x = 0.$$

a. Find the largest interval I that contains the point $t = 1$ and on which (**) is normal.

Solution: $(0, \infty)$

b. Show that the functions t^3 and t^4 are solutions on this interval I.

Solution: Plug them in:
$t^2 \cdot 3 \cdot 2t - 6t \cdot 3t^2 + 12t^3 = 0$ ✓ and $t^2 \cdot 4 \cdot 3t^2 - 6t \cdot 4t^3 + 12t^4 = 0$ ✓.

6. (10 points)

a. Evaluate the determinant $\det \begin{pmatrix} 0 & 1 & 6 & 9 & 1 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

b. Consider the linear equations

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 1 \\ 2x_2 + x_3 \qquad &= 2 \\ 3x_3 + x_4 &= 3 \\ x_1 + x_2 + x_3 + 4x_4 &= 4 \end{aligned}$$

Use Cramer's rule to solve for x_4 *in terms of determinants*.
DO NOT EVALUATE THE DETERMINANTS.

Solution: $x_4 = \dfrac{\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}}$.

7. (10 points) What is the Wronskian of $h_1(t) = te^t$ and $h_2(t) = t^2e^t$ at $t = 0$?

Solution: $W[te^t, t^2e^t](0) = \begin{vmatrix} 0 & 0 \\ * & * \end{vmatrix} = 0$. (Generally, $W[te^t, t^2e^t] = \begin{vmatrix} te^t & t^2e^t \\ (t+1)e^t & (t^2+2t)e^t \end{vmatrix}$.)

8. (10 points) A savings account pays 3% interest per year, compounded continuously. In addition, the income from another investment is credited to the account continuously, at the rate of \$700 per year. Set up a differential equation to model this account.

Solution: $x' = 0.03x + 700$.

9. (10 points) Find the solution of $x'' = te^t$ with $x(0) = x'(0) = 0$.

Solution: Using integration by parts or the pattern observed in the Wronskian above one finds $x'(t) = (t-1)e^t + C_1$ (with $0 = x'(0) = -1 + C_1$, so $C_1 = 1$) and then $x(t) = (t-2)e^t + t + C_2$ (with $0 = x(0) = -2 + C_2$, so $C_2 = 2$), which gives the solution $x(t) = (t-2)e^t + t + 2$.