Mathematics 38 **Differential Equations**
Exam I **12:00–1:20, September 29, 2008**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.
Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (5 points) Check that $1/t$ and $1/t^2$ are solutions of $(t^2D^2 + 4tD + 2)x = 0$ for $t > 0$ and decide whether those solutions form a complete collection of solutions.

Solution:
For $1/t$ we get $(t^2D^2 + 4tD + 2)t^{-1} = t^2(-1)(-2)t^{-3} + 4t(-1)t^{-2} + 2t^{-1} = (2 - 4 + 2)t^{-1} = 0$ for all $t > 0$.
For $1/t^2$ we get $(t^2D^2 + 4tD + 2)t^{-2} = t^2(-2)(-3)t^{-4} + 4t(-2)t^{-3} + 2t^{-2} = (6 - 8 + 2)t^{-2} = 0$ for all $t > 0$.
They form a complete collection because the Wronskian is $\det \begin{pmatrix} t^{-1} & t^{-2} \\ -t^{-2} & -2t^{-3} \end{pmatrix} = (-2 + 1)t^{-4} \neq 0$ for $t > 0$.

2. (5 points) Make a *simplified* guess for a particular solution of

$$(D - 1)(D^2 + 1)^3(D + 2)x = t^2e^t + e^{-t}\sin 3t + t.$$

Do not try to determine the coefficients!

Solution: $c_1te^t + c_2t^2e^t + c_3t^3e^t + c_4e^{-t}\sin 3t + c_5e^{-t}\cos 3t + c_6 + c_7t$.

3. (5 points) Determine whether the system

$$\begin{aligned} x - y + 3z &= a \\ 2x \quad - \quad z &= b \\ x + y - 3z &= c \end{aligned}$$

has solutions for all values of the right-hand side.

Solution: $\det \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix} = \det \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} - \det \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = (-6+1) - (-1-6) = 2 \neq 0$, so by Cramer's test the system has a (unique) solution for all values of the right-hand side.

(75 points) In problems **4–9**

- **a.** State the order of the differential equation.
- **b.** If the differential equation is of first order, state whether it is separable or not.
- **c.** If the differential equation is separable, find the general solution and **skip parts d)–f)**.
- **d.** State whether the differential equation is linear. **If it is not, skip parts e)–f)**.
- **e.** If the differential equation is linear find the largest interval containing 1 on which the differential equation is normal; state whether the differential equation is homogeneous or nonhomogeneous, and whether it has constant coefficients or not.
- **f.** If the differential equation is linear, find the general solution.

4. $t^3x' = x^3$.

Solution: a. 1, b. yes, c. $x = 0$ or $\dfrac{dx}{x^3} = \dfrac{dt}{t^3}$, so $\dfrac{1}{x^2} = \dfrac{1}{t^2} + c = \dfrac{1 + ct^2}{t^2}$ and $x(t) = \pm\dfrac{t}{\sqrt{1 + ct^2}}$.

5. $x'x^2 + t = 0$.

Solution: a. 1, b. yes, c. $x^2\,dx = -t\,dt$, so $x^3/3 = -t^2/2 + c$ and $x = (-\frac{3}{2}t^2 + c)^{1/3}$.

6. $9x'' - 12x' + 4x = 0.$

Solution: a. 2, d. yes, e. $\mathbb{R}$, homogeneous, constant coefficients, f. $c_1 e^{2t/3} + c_2 t e^{2t/3}$.

7. $(D^2 + 1)^2 x = 0.$

Solution: a. 4, d. yes, e. $\mathbb{R}$, homogeneous, constant coefficients, f. $c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$.

8. $x'' + x = \sec t.$

Solution: a. 2, d. yes, e. $(0, \pi/2)$, nonhomogeneous, constant coefficients. f. The general solution of the associated homogeneous differential equation is generated by $h_1(t) = \cos t$ and $h_2(t) = \sin t$. Variation of parameters gives

$$\begin{aligned} c_1' \cos t + c_2' \sin t &= 0 \\ -c_1' \sin t + c_2' \cos t &= \sec t. \end{aligned}$$

This implies $c_1' = -\dfrac{\sin t}{\cos t}$ and $c_2' = 1$. Therefore, $c_1(t) = \int \frac{-\sin t}{\cos t}\, dt = \ln \cos t$ and $c_2'(t) = t$. Therefore, the general solution is $x(t) = c_1 \cos t + c_2 \sin t + \cos t \cdot \ln \cos t + t \sin t$.

9. $x' + \sin x + t = 0.$

Solution: a. 1, b. no, d. no.

10. (10 points) Decide whether the functions t^5, $|t|^5$ are linearly independent on $(-\infty, \infty)$. *Justify your conclusion.*

Solution: They are linearly independent: If $c_1 t^5 + c_2 |t|^5 = 0$ for all t then for $t = 1$ we get $c_1 + c_2 = 0$ and for $t = -1$ we get $-c_1 + c_2 = 0$. Adding and subttracting these two equations gives $c_1 = c_2 = 0$.

END OF EXAMINATION