

No calculators, notes, or books are allowed. Please make sure all electronic devices you carry are turned off and put away out of sight.

*Remember to sign your blue book.* With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (3 points each, no partial credit) For each of the differential equations below determine the order, determine whether the differential equation is linear, and if so, whether it is homogeneous.

a.  $t^4 \frac{d^3 x}{dt^3} + t \frac{dx}{dt} - t - x^7 = 0$

**Solution:** Third order, nonlinear because of  $x^7$  term.

b.  $\frac{dx}{dt} + \frac{d^7 x}{dt^7} = x + t^9$

**Solution:** Seventh order, linear, non-homogeneous.

c.  $\left(\frac{dx}{dt}\right)^5 + \frac{d^4 x}{dt^4} - t^3 x + t^7 = 0$

**Solution:** Fourth order, nonlinear

d.  $x'x''' = x^4x'' + t^5x'$

**Solution:** Third order, nonlinear

2. (3 points each, no partial credit) Find all real values of  $\alpha$  for which the given function is a solution of the given differential equation.

a.  $x = \alpha, \quad \frac{d^7 x}{dt^7} + \sin t \frac{dx}{dt} + x - 4 = 0$

**Solution:** Substitution of  $x = \alpha$  gives  $\alpha - 4 = 0$ , or  $\alpha = 4$ .

b.  $x = \frac{\alpha}{t^2 + 1}, \quad \frac{dx}{dt} + 2tx^2 = 0$

**Solution:** Substitution of  $x = \frac{\alpha}{t^2 + 1}$  gives  $\frac{-2\alpha t}{(t^2 + 1)^2} + \frac{2\alpha^2 t}{(t^2 + 1)^2} = 0$ ,

or  $\frac{2(\alpha^2 - \alpha)t}{(t^2 + 1)^2} = 0$ , so  $\alpha^2 - \alpha = \alpha(\alpha - 1) = 0$ , or  $\alpha = 0, 1$ .

c.  $x = e^{\alpha t}, \quad xx' = e^{2t}$

**Solution:** The derivative of  $x$  is  $x' = \alpha e^{\alpha t}$ , so substitution gives  $\alpha e^{2\alpha t} = e^{2t}$ . Thus,  $\alpha = 1$ .

3. (1 point each) For each of the following differential equations state whether it is normal on  $0 < t < 2$ .

a.  $(t - 1) \frac{d^7 x}{dt^7} + 4 \frac{dx}{dt} - 5x = 3t$

**Solution:**  $(t - 1) = 0$  at  $t = 1$ , so the coefficient of the highest-order term is equal to 0 on  $0 < t < 2$ . The ODE is not normal.

b.  $3 \frac{dx}{dt} - 5x = \csc \pi t$

**Solution:**  $\csc \pi t = \frac{1}{\sin \pi t}$ , but  $\sin(\pi \cdot 1) = 0$  and, so, the right-hand side is not continuous for  $0 < t < 2$ . The ODE is not normal.

c.  $t \frac{dx}{dt} + e^t x = \sqrt{t}$

**Solution:**  $t, e^t$ , and  $\sqrt{t}$  are all continuous on  $0 < t < 2$ , and the coefficient of the highest-order term is never 0 on  $0 < t < 2$ , so the ODE is normal.

d.  $t \sin t \frac{dx}{dt} + \pi x = \ln t$

**Solution:**  $t \sin t$  and  $\ln t$  are continuous on  $0 < t < 2$  and the coefficient of the highest-order term is never 0 on  $0 < t < 2$ , so the ODE is normal.

4. (10 points)

a. Evaluate the determinant  $\det \begin{pmatrix} 0 & 1 & 6 & 9 & 1 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix}$ .

**Solution:**  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ .

b. Consider the linear equations

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_2 + x_3 &= 2 \\ 3x_3 + x_4 &= 3 \\ x_1 + x_2 + x_3 + 4x_4 &= 4 \end{aligned}$$

Use Cramer's rule to solve for  $x_4$  in terms of determinants.

**DO NOT EVALUATE THE DETERMINANTS.**

**Solution:**  $x_4 = \frac{\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}}$ .

5. (5 points) What is the Wronskian of  $h_1(t) = te^t$  and  $h_2(t) = t^2e^t$  at  $t = 0$ ?

**Solution:**  $W[te^t, t^2e^t](0) = \begin{vmatrix} 0 & 0 \\ * & * \end{vmatrix} = 0$ . (Generally,  $W[te^t, t^2e^t] = \begin{vmatrix} te^t & t^2e^t \\ (t+1)e^t & (t^2+2t)e^t \end{vmatrix}$ .)

6. (10 points) Determine whether the functions  $\sin t$  and  $|\sin t|$  are linearly independent on  $-\infty < t < \infty$ . Explain! (Hint: Computing the Wronskian may *not* be the best approach.)

**Solution:** Set  $c_1 \sin t + c_2 |\sin t| = 0$ . If the functions are linearly independent, then the only values of  $c_1$  and  $c_2$  such that this is always true for  $-\infty < t < \infty$  should be  $c_1 = c_2 = 0$ .

Taking  $t = \frac{\pi}{2}$  gives  $c_1 + c_2 = 0$ . Taking  $t = -\frac{\pi}{2}$  gives  $-c_1 + c_2 = 0$ .

The only values of  $c_1$  and  $c_2$  that satisfy these equations are  $c_1 = c_2 = 0$ , so the two functions must be linearly independent.

7. (15 points) Solve the initial-value problem  $\frac{dx}{dt} - tx = t$ ,  $x(0) = \frac{1}{2}$ .

**Solution:** Rewriting the ODE, we have  $\frac{dx}{dt} = tx + t = (x+1)t$ . Separating variables gives  $\frac{dx}{x+1} = t dt$  for  $x \neq -1$  (but  $x = -1$  doesn't satisfy the initial condition, so we can discard this solution). Integrating then gives  $\ln|x+1| = \frac{1}{2}t^2 + c$ . Exponentiating gives  $x+1 = ke^{t^2/2}$ , or

$$x(t) = ke^{t^2/2} - 1.$$

Applying the initial condition gives  $x(0) = k - 1 = \frac{1}{2}$ , or  $k = \frac{3}{2}$ , with  $x(t) = \frac{3}{2}e^{t^2/2} - 1$ .

**Alternate solution:** Take homogenous part of the equation,  $\frac{dx}{dt} - tx = 0$  and solve using separation of variables.

This now gives  $\frac{dx}{x} = t dt$ , which, when integrated, yields  $\ln|x| = \frac{t^2}{2} + c$ . So, the solution to the homogeneous problem is  $x = ke^{t^2/2}$ . Now, make the variation of parameters ansatz, that  $x = k(t)e^{t^2/2}$ , and substitute into the original ODE.

This gives  $k'(t)e^{t^2/2} = t$ , or  $k'(t) = te^{-t^2/2}$ . This is easily integrated by making the change of variables,  $u = \frac{t^2}{2}$ , so that  $du = t dt$ , and we find that  $k(t) = -e^{-t^2/2} + c$ . Plugging this in gives

$$x(t) = \left(-e^{-t^2/2} + c\right) e^{t^2/2} = ce^{t^2/2} - 1.$$

Plugging in the initial condition gives  $c = 3/2$ , as in the other solution.

8. (10 points) Find the general solution of  $\frac{d^4x}{dt^4} - x = 0$ .

**Solution:** First find the characteristic polynomial. Writing the ODE as  $(D^4 - 1)x = 0$ , we can write the operator as  $P(D) = D^4 - 1$ , giving  $P(r) = r^4 - 1$  as the characteristic polynomial. Factoring this gives  $P(r) = (r^2 - 1)(r^2 + 1) = (r - 1)(r + 1)(r^2 + 1)$ . So,  $P(r)$  has four roots,  $r = \pm 1, \pm i$ , each with multiplicity 1. The general solution is then

$$x(t) = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t.$$

9. (15 points) Solve the initial-value problem  $x^{(5)} - 2x^{(4)} + x^{(3)} = 0$  with  $x(0) = 1, x'(0) = 2, x''(0) = 6, x'''(0) = 0$  and  $x^{(4)}(0) = 0$ . (Here  $x^{(n)}$  denotes the  $n$ th derivative.)

**Solution:** Again writing the ODE in operator notation, we have  $(D^5 - 2D^4 + D^3)x = 0$ , so the characteristic polynomial is  $P(r) = r^5 - 2r^4 + r^3$ . Factoring this gives  $P(r) = r^3(r - 1)^2$ . So,  $P(r)$  has two roots,  $r = 0$  with multiplicity 3, and  $r = 1$  with multiplicity 2. This gives the general solution as

$$x(t) = c_1 + c_2t + c_3t^2 + c_4e^t + c_5te^t.$$

Taking derivatives, we have

$$\begin{aligned} x(t) &= c_1 + c_2t + c_3t^2 + c_4e^t + c_5te^t \\ x'(t) &= c_2 + 2c_3t + c_4e^t + c_5(t+1)e^t \\ x''(t) &= 2c_3 + c_4e^t + c_5(t+2)e^t \\ x'''(t) &= c_4e^t + c_5(t+3)e^t \\ x^{(4)}(t) &= c_4e^t + c_5(t+4)e^t \end{aligned}$$

Initial conditions  $x'''(0) = 0$  and  $x^{(4)}(0) = 0$  translate into  $\begin{matrix} c_4 + 3c_5 = 0 \\ c_4 + 4c_5 = 0 \end{matrix}$ , which has unique solution  $c_4 = c_5 = 0$ . To match the other three initial conditions, we need  $c_1 = 1$  (so that  $x(0) = 1$ ),  $c_2 = 2$  (so that  $x'(0) = 2$ ), and  $c_3 = 3$  (so that  $x''(0) = 6$ ). This gives

$$x(t) = 1 + 2t + 3t^2.$$

10. (10 points) Suppose  $f(t)$  is a continuous function. Give a solution of the initial-value problem  $x' + f(t)x = 0, x(1) = 0$ . (Hint: Think before applying standard techniques.)

**Solution:** By inspection,  $x(t) = 0$  (for all  $t$ ) is a solution of the ODE (since  $\frac{dx}{dt} = 0$ ), and it satisfies the initial condition. (Note that the Theorem about existence and uniqueness applies, so any different answer is wrong.)

One could get this by separation of variables (not recommended): We rewrite the equation as  $\frac{dx}{dt} = -f(t)x$ , and separate variables to give  $\frac{dx}{x} = -f(t)dt$  (if  $x \neq 0$ ). Integrating (and forgetting about absolute values), we get  $\ln x = -\int f(t)dt + C$ , so  $x = e^{C - \int f(t)dt} = Ae^{-\int f(t)dt}$  for some constant  $A$  (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we can take  $A = 0$ , which gives  $x(t) = 0$  for all  $t$ .