

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!

Remember to sign your blue book.

With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (5 points, answer only, no partial credit) Can the system below be solved by Cramer's rule?

$$\begin{aligned}x_1 + x_2 + x_3 &= 0 \\2x_1 + 3x_2 + x_3 &= 0 \\-x_1 - 3x_2 + x_3 &= 0\end{aligned}$$

**Solution:** No, the determinant of the matrix of coefficients is zero.

2. (5 points, answer only, no partial credit) Consider the following system of equations:

$$\begin{aligned}x_1 + x_2 + x_3 &= 0 \\2x_1 + 3x_2 + x_3 &= 0 \\-x_1 - 3x_2 + x_3 &= 0\end{aligned}$$

Choose one answer: The system has

**a.** a unique solution, **b.** no solution, **c.** more than 1 solution. **d.** None of the above.

**Solution:** **c.** **d.** is clearly impossible, **a.** is impossible because the previous problem shows that Cramer's test gives a zero determinant, and **b.** is not true because  $x_1 = x_2 = x_3 = 0$  is an obvious solution.

3. (5 points, answer only, no partial credit) Choose one answer.

$$\det \begin{pmatrix} 5 & 1 & \sin t & t^2 + 3 & 1 \\ 0 & 4 & e^t & e^t & 0 \\ 0 & 0 & 3 & \ln t & 8 \\ 0 & 0 & 0 & 2 & \sqrt{t} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

**a.** 5, **b.**  $2\sqrt{t}$ , **c.** 120, **d.**  $120 - \sin t - \frac{1}{4}e^t - \frac{1}{4}(t^2 + 3)e^t$ . **e.** None of the above.

**Solution:** **c.**; this is easy because the matrix is triangular.

4. (5 points, answer only, no partial credit) Choose one answer.

The functions  $e^t$ ,  $e^{-t}$ ,  $e^{t+1}$  are

**a.** linearly dependent **b.** linearly independent. **c.** None of the above.

**Solution:** **a.** because  $e \cdot e^t + 0 \cdot e^{-t} - 1 \cdot e^{t+1} = 0$  for all  $t$ .

5. (5 points, answer only, no partial credit) Consider the differential equation

$$t^4 x'' - 2xx' = 0.$$

Choose every function from the following that is a solution of this differential equation.

**a.** 0, **b.** 1, **c.**  $t$ , **d.**  $t^2$ , **e.**  $t^3$ , **f.**  $t^4$ , **g.** None of the above.

**Solution:** **a.**, **b.**, **e.**: Plug them in . . .

6. (5 points, answer only, no partial credit) Choose the correct answer:

$$(D - 1)^3(t^4 e^t) =$$

**a.**  $(4t^3 + 4)e^t$ , **b.**  $4t^3 e^t$ , **c.**  $(4t^3 + 24t^2 + 24t)e^t$ , **d.**  $(-t^4 + 8t^3 + 12t^2 - 24t)e^t$ , **e.**  $12t^2 e^t$ , **f.**  $24te^t$ , **g.** None of the above.

**Solution:** **f.**: This is easiest by the exponential shift.

7. (4 points, answer only, no partial credit) Consider the differential equation

$$(t - 1)x' = -x.$$

Choose all correct answers from below. For the initial value  $(t_0, \alpha) = (1, 0)$ :

- a. The theorem about existence and uniqueness of solutions applies, and the differential equation has a unique solution with this initial value.
- b. The theorem about existence and uniqueness of solutions applies, and the differential equation has no solution with this initial value.
- c. The theorem about existence and uniqueness of solutions applies, and the differential equation has more than one solution with this initial value.
- d. The theorem about existence and uniqueness of solutions does not apply, and the differential equation has a unique solution with this initial value.
- e. The theorem about existence and uniqueness of solutions does not apply, and the differential equation has no solution with this initial value.
- f. The theorem about existence and uniqueness of solutions does not apply, and the differential equation has more than one solution with this initial value.
- g. None of the above.

**Solution: d.:** This was on the homework.

8. (4 points, answer only, no partial credit) Again consider the differential equation  $(t - 1)x' = -x$  but now with the initial value  $(t_0, \alpha) = (0, 1)$ . From the statements **a.-g.** in the previous problem choose all that are correct for this new initial value.

**Solution: a.:**  $-x/(t - 1)$  is continuous at  $(0, 1)$ .

9. (4 points, answer only, no partial credit) Yet again consider the differential equation  $(t - 1)x' = -x$  but now with the initial value  $(t_0, \alpha) = (1, 1)$ . From the statements **a.-g.** in Problem 7. choose all that are correct for this initial value.

**Solution: e.:** Insert  $t = 1$  and  $x = 1$  into the differential equation to get  $0 = 1$ , which is impossible.

10. (10 points, answer only, no partial credit) Consider the differential equation

$$(D - 1)^2x = e^t + t.$$

If one uses the method of undetermined coefficients, then the correct *simplified guess* for a particular solution is

- a.  $A + Bt^2e^t$ ,    b.  $A + Bt + Ce^t$ ,    c.  $A + Bt + Cte^t$ ,    d.  $Ae^t + Bte^t$ ,    e.  $At + Ce^t$ ,
- f.  $A + Bt + Ct^2e^t$ ,    g.  $At + B \sin t + C \cos t$ ,    h.  $A + Bt + C \sin t + D \cos t$ ,    i.  $A + Bt^2e^t + C \sin t + D \cos t$ ,
- j.  $A + Bt + Ce^t + D \sin t + E \cos t$ ,    k.  $A + Bt + Cte^t + D \sin t + E \cos t$ ,    l.  $At + Be^t + C \sin t + D \cos t$ ,
- m.  $A + Bt + Ct^2e^t + D \sin t + E \cos t$ ,    n. None of the above.

**Solution: f.**

11. (15 points) Solve the initial-value problem  $\frac{dx}{dt} - tx = t$ ,  $x(0) = \frac{1}{2}$ .

**Solution:** Write  $x' = t(x + 1)$  and separate variables to get  $x(t) = Ae^{t^2/2} - 1$ . Then  $x(0) = 1/2$  gives  $A = 3/2$ , so  $x(t) = \frac{3}{2}e^{t^2/2} - 1$ .

12. (10 points) Find the general solution of  $\frac{d^4x}{dt^4} - x = 0$ .

**Solution:** The roots of  $r^4 - 1$  are  $\pm 1$  and  $\pm i$ , so the general solution is  $c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t$ .

13. (5 points) Determine constants  $k_1$  and  $k_2$  such that  $p(t) = k_1t \cos 2t + k_2t \sin 2t$  is a solution of

$$(D^2 + 4)x = \sin 2t.$$

**Solution:**  $(D^2 + 4)p(t) = -4k_1 \sin 2t + 4k_2 \cos 2t$ , so  $k_1 = -1/4$ ,  $k_2 = 0$ .

14. (5 points) Solve the following system of equations for  $x$  only:

$$\begin{aligned}x + 2y - 3z &= 1 \\2x + 2y + 3z &= 2 \\x - y + z &= 0\end{aligned}$$

**Solution:** Cramer's rule gives  $x = \frac{\det \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}} = \frac{5 - 2(2 - 3)}{5 - 2(2 - 3) + 12} = \frac{7}{19}$

15. (7 points) Suppose  $f(t)$  is a continuous function. Give a solution of the initial-value problem  $x' + f(t)x = 0$ ,  $x(1) = 0$ .  
(Hint: Think before applying standard techniques.)

**Solution:** By inspection,  $x(t) = 0$  (for all  $t$ ) is a solution of the ODE (since  $\frac{dx}{dt} = 0$ ), and it satisfies the initial condition. (Note that the Theorem about existence and uniqueness applies, so any different answer is wrong.)

One could get this by separation of variables (not recommended): We rewrite the equation as  $\frac{dx}{dt} = -f(t)x$ , and separate variables to give  $\frac{dx}{x} = -f(t)dt$  (if  $x \neq 0$ ). Integrating (and forgetting about absolute values), we get  $\ln x = -\int f(t)dt + C$ , so  $x = e^{C - \int f(t)dt} = Ae^{-\int f(t)dt}$  for some constant  $A$  (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we can take  $A = 0$ , which gives  $x(t) = 0$  for all  $t$ .