Exam 1 2/9/2009

#1

3 (a) Linear yes

3 (b) Normal yes

3 (c) $\mathbb{R}$

Exponential Shift:

$$P(D)\left(ue^{\lambda t}\right) = e^{\lambda t}\, P(D+\lambda)\, u$$

3 (d) $\left(D^2+3D+2\right) t^2 e^{-t}$

$$= e^{-t}\left[(D-1)^2 + 3(D-1) + 2\right] t^2$$

12 $$= e^{-t}\left(D^2 + D\right) t^2$$

$$= e^{-t}\left(2 + 2t\right)$$

$$= \boxed{2(t+1)e^{-t}}$$

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$$f(t,x) = -\frac{x}{t-2}$$

#2 (a) Not continuous at (2, 1)

(b) $f_x(t,x) = -\frac{1}{t-2}$ Not continuous at (2, 1)

~~(c) No guarantee of existence or uniqueness~~ Next page

~~(d) $\frac{dx}{dt} = \frac{x}{t-2}$ → $\ln|x| = \ln|t-2| + c$~~

~~$\frac{dx}{x} = \frac{dt}{t-2}$ $x = k(t-2)$~~

~~No solution satisfying $x(2) = 1$~~

~~(e) (2, 1) is not within a rectangle where f and f_x are continuous.~~

~~Existence is not guaranteed.~~

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#2 (c) $\frac{dx}{dt} = -\frac{x}{t-2}$

$f(t,x) = -\frac{x}{t-2}$ $\quad f_x(t,x) = -\frac{1}{t-2}$

not continuous at $t=2$

E & U not satisfied at $x(2)=1$

(d)

$\frac{dx}{x} = -\frac{dt}{t-2}$, $x \neq 0$

$\ln|x| = -\ln|t-2| + c$

$x(t) = \frac{k}{t-2}$

No solution through $(2,1)$

(e) Rectangles of continuity for f and f_x include

$-\infty < t < 2$
$-\infty < x < \infty$

and

$2 < t < \infty$
$-\infty < x < \infty$

E & U theorem does not apply at $(2,1)$ because the point is not in either rectangle.

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#3 (a) Try $x = t^\alpha$ in

$$(t^2D^2 - 3tD + 2)x = 0$$

$$\alpha(\alpha-1)t^\alpha - 3\alpha t^\alpha + 2t^\alpha = 0$$

+9

$$\alpha(\alpha-1) - 3\alpha + 2 = 0$$

$$\alpha^2 - 4\alpha + 2 = 0$$

$$\alpha = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2} \qquad \boxed{\alpha = 2 \pm \sqrt{2}}$$

(b) Try $x = \alpha t^2 + 2\alpha t + 2\alpha$ in

$$(D-1)x = t^2$$

+9

$$2\alpha t + 2\alpha - \alpha t^2 - 2\alpha t - 2\alpha = t^2$$

$$-\alpha t^2 = t^2 \Rightarrow \boxed{\alpha = -1}$$

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#4 Solve $x' - x = e^t \sin t$ (N)

(H) $x' - x = 0$

$$\frac{dx}{x} = dt$$

$$\ln|x| = t + c \qquad x = ke^t$$

(V) Try $x = k(t)e^t$ in (N)

$$x' = k'e^t + ke^t$$

$$x' - x = k'e^t = e^t \sin t$$

$$k' = \sin t \qquad k(t) = -\cos t + c$$

$$x = (c - \cos t)e^t$$

$$\boxed{x(t) = ce^t - e^t \cos t}$$

$c = c$

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#5

(a)
$$\begin{vmatrix} e^t & \sin t & \cos t \\ e^t & \cos t & -\sin t \\ e^t & -\sin t & -\cos t \end{vmatrix} = e^t \begin{vmatrix} 1 & \sin t & \cos t \\ 1 & \cos t & -\sin t \\ 1 & -\sin t & -\cos t \end{vmatrix}$$

$$= e^t \left(\begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} - \begin{vmatrix} \sin t & \cos t \\ -\sin t & -\cos t \end{vmatrix} + \begin{vmatrix} \sin t & \cos t \\ \cancel{\sin t} \; \cos t & \cancel{\cos t} \; -\sin t \end{vmatrix} \right)$$

$$= e^t \left[(-\cos^2 t - \sin^2 t) - 0 + (-\sin^2 t - \cos^2 t) \right]$$

$$= -2e^t$$

(b)
$$(D^2+1)\sin t = 0$$
$$(D^2+1)\cos t = 0$$
$$(D-1)\,e^t = 0$$

e^t, $\sin t$, $\cos t$ are solutions to the 3rd order, linear ODE

(H) $$(D-1)(D^2+1)\,x = 0$$

By Part (a), the three functions are linearly independent.

$\therefore$ $$x = c_1 e^t + c_2 \sin t + c_3 \cos t$$

is the general solution to (H).

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#6

(a) $[D^3-2D^2+D]x=0$ has characteristic polynomial $p(r)=r^3-2r^2+r$

$$= r(r^2-2r+1)$$
$$= r(r-1)^2$$

with roots $r=0, 1, 1$

$$\boxed{x(t)= c_1+c_2e^t+c_3te^t}$$

(b) $x' = (c_2+c_3)e^t+c_3te^t$

$x'' = (c_2+2c_3)e^t+c_3te^t$

$x''(0)=1 \Rightarrow c_2+2c_3=1$

$x'(0)=0 \Rightarrow c_2+c_3=0$

$x(0)=0 \Rightarrow c_1+c_2=0$

$c_1=1 \quad c_2=-1 \quad c_3=1$

$$\boxed{x(t)= 1-e^t+te^t}$$