

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!

Remember to sign your blue book.

With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

Please put the answers to problems 1–10 on the blue book cover in the corresponding box, as shown here:

1	Yes
2	w.
3	e.
4	c.
5	a.n.
6	cos(t)
7	high
8	Yes
9	
10	
T	

1. (5 points, answer only, no partial credit) Can the system below be solved by Cramer's rule?

$$\begin{aligned} x+2y+ z &= 0 \\ 3x+6y+3z &= 0 \\ -x &+ z = 0 \end{aligned}$$

Solution: No, the second equation is 3 times the first, so the determinant of the coefficient matrix is zero.

2. (5 points, answer only, no partial credit) Consider the following system of equations:

$$\begin{aligned} x+2y+ z &= 0 \\ 2x- y+3z &= 0 \\ 2x-6y+4z &= 0 \end{aligned}$$

Choose one answer: The system has

- a.** a unique solution, **b.** no solution, **c.** more than 1 solution. **d.** None of the above.

Solution: **c.;** **d.** is clearly impossible, **a.** is impossible because Cramer's test gives a zero determinant, and **b.** is not true because $x = y = z = 0$ is an obvious solution.

3. (5 points, answer only, no partial credit) Choose one answer.

$$\det \begin{pmatrix} t & 1 & \sin t & \ln t & 1 \\ t & 4 & \cos t & e^t & 3 \\ 0 & 0 & \tan t & t & \sqrt{t} \\ 0 & 0 & 0 & t^2 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

- a.** $4t^3 \tan t$, **b.** $t \sin t \ln t$, **c.** $3t^3 \tan t - 3t^2$, **d.** None of the above.

Solution: **d.;** subtract row 1 from row 2 first to make the matrix triangular; the determinant is $3t^3 \tan t$.

4. (5 points, answer only, no partial credit) Consider the differential equation

$$(t - 3) \frac{dx}{dt} = 2x.$$

Choose all correct answers from below. For the initial value $(t_0, \alpha) = (3, 1)$:

- a.** The theorem about existence and uniqueness of solutions applies, and the differential equation has a unique solution with this initial value.
- b.** The theorem about existence and uniqueness of solutions applies, and the differential equation has no solution with this initial value.
- c.** The theorem about existence and uniqueness of solutions applies, and the differential equation has more than one solution with this initial value.
- d.** The theorem about existence and uniqueness of solutions does not apply, and the differential equation has a unique solution with this initial value.
- e.** The theorem about existence and uniqueness of solutions does not apply, and the differential equation has no solution with this initial value.
- f.** The theorem about existence and uniqueness of solutions does not apply, and the differential equation has more than one solution with this initial value.
- g.** None of the above.

Solution: **e.;** According to the differential equation, $t = 3$ implies $x = 0$, which is inconsistent with the initial data.

5. (5 points, answer only, no partial credit) Again consider the differential equation $(t - 3)\frac{dx}{dt} = 2x$ but now with the initial value $(t_0, \alpha) = (0, 1)$. From the statements **a.–g.** in Problem 4 choose all that are correct for these new initial data.

Solution: a.: $2x/(t - 3)$ is continuous at $(0, 1)$.

6. (5 points, answer only, no partial credit) Yet again consider the differential equation $(t - 3)\frac{dx}{dt} = 2x$ but now with the initial value $(t_0, \alpha) = (3, 0)$. From the statements **a.–g.** in Problem 4 choose all that are correct for these initial data.

Solution: f.: Separation of variables gives solutions $x(t) = k(t - 3)^2$ for any k .

7. (5 points, answer only, no partial credit) Compute

$$(D + 3)^4(t^5 e^{-3t})$$

Solution: $(D + 3)^4(t^5 e^{-3t}) = e^{-3t} D^4 t^5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot t \cdot e^{-3t}$ by the exponential shift.

8. (10 points, answer only, no partial credit) Consider the differential equation

$$D(D - 1)^2 x = 1 + te^t.$$

If one uses the method of undetermined coefficients, then the correct *simplified guess* for a particular solution is

- a.** $A + Bt^2 e^t$, **b.** $A + Bt + Ce^t$, **c.** $A + Bt + Cte^t$, **d.** $Ae^t + Bte^t$, **e.** $At + Ce^t$, **f.** $A + Bt + Ct^2 e^t$,
g. $A + Bt + Cte^t + Dt^2 e^t$, **h.** $At + Bt^2 e^t + Ct^3 e^t$, **i.** $At + Bte^t + Ct^2 e^t$, **j.** $At + B \sin t + C \cos t$,
k. None of the above.

Solution: h.: Both parts of the annihilator $D(D - 1)^2$ match terms on the left-hand side.

9. (5 points, answer only, no partial credit) Choose one answer.

The functions e^{3t} , t , e^{-2t} , e^{3t+1} are

- a.** linearly dependent **b.** linearly independent **c.** None of the above.

Solution: a. because $e \cdot e^{3t} + 0 \cdot t + 0 \cdot e^{-2t} - 1 \cdot e^{3t+1} = 0$ for all t . (Computing the Wronskian instead is not a great move—it turns out to be zero, but that does not always imply linear independence, so one is stuck here.)

10. (5 points, answer only, no partial credit) Consider the differential equation

$$t^2 x'' + tx' - x = 0.$$

Choose every function from the following that is a solution of this differential equation.

- a.** 0, **b.** 1, **c.** t , **d.** $1/t$, **e.** t^2 , **f.** t^3 , **g.** $\frac{t^2 + 1}{t}$, **h.** e^t , **i.** e^{-t} , **j.** $3te^t \sin t$,
k. None of the above.

Solution: a., c., d., g.: Plug in 0, t , $1/t$ to see that these work; t and $1/t$ are linearly independent, hence generate the general solution, so $\frac{t^2 + 1}{t} = t + \frac{1}{t}$ is a solution but none of the others are (no need to plug them in!).

11. (12 points) Find the general solution of $(D^2 + 1)^4 x = 0$.

Solution: The roots of $(r^2 + 1)^4$ are $\pm i$ with multiplicity 4, so the general solution is $c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + c_5 t^2 \cos t + c_6 t^2 \sin t + c_7 t^3 \cos t + c_8 t^3 \sin t$.

12. (20 points) Solve the initial-value problem

$$(D^2 + 1)x = \sin 3t, \quad x(0) = 0, \quad x'(0) = 1.$$

Solution: Using variation of parameters here would be foolish. The method of undetermined coefficients gives the simplified guess $A \cos t + B \sin t$, plugging in gives the general solution $c_1 \cos t + c_2 \sin t - \frac{1}{8} \sin 3t$. $x(0) = 0$ implies $c_1 = 0$; then $x'(0) = 1$ implies $c_2 - \frac{3}{8} = 1$, hence $c_2 = \frac{11}{8}$ and the solution is $x(t) = \frac{11}{8} \sin t - \frac{1}{8} \sin 3t$.

13. (8 points) Solve the following system of equations for x only:

$$\begin{aligned}x + y + z &= 1 \\3x + 2y - z &= 0 \\x - y + z &= 0\end{aligned}$$

Solution: To pretty this up a little we can subtract the last equation from the first. Then Cramer's rule gives

$$x = \det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} / \det \begin{pmatrix} 0 & 2 & 0 \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{2 - 1}{-2(3 + 1)} = -\frac{1}{8}.$$

14. (5 points) Demonstrate that the system

$$\begin{aligned}3x_1 + x_2 - x_3 + 4x_4 + 7x_5 &= 0 \\x_1 + x_2 + x_3 - x_4 + x_5 &= 0 \\x_1 - x_3 + x_5 &= 0 \\2x_1 + x_2 + x_4 + 2x_5 &= 0 \\x_1 + 3x_2 + x_3 - x_4 + 5x_5 &= 0\end{aligned}$$

has a solution.

Solution: By inspection $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ is a solution. (One could compute the determinant of the coefficient matrix, but that is not time well spent: If it is zero, then we gain no information that is useful for the question; otherwise we obtain uniqueness, but that wasn't asked for.)