

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (16 points) Consider the equation

$$(t + 1)x' + tx = 0$$

- a. Is it separable?

Solution: Yes

- b. Is it linear?

Solution: Yes

- c. Is it homogeneous?

Solution: Yes

- d. Give the largest interval containing 0 where the equation is normal.

Solution: $(-1, \infty)$

- e. Write the equation in standard form.

Solution: $x' + \frac{t}{t+1}x = 0$

- f. Solve the equation.

Solution: Separation of variables: $\ln x = -\int \frac{t}{t+1} dt = -t + \ln|t+1| + C$, so $x(t) = A(t+1)e^{-t}$.

- g. Find a solution with $x(0) = 2$.

Solution: $2 = x(0) = A$ gives $x(t) = 2(t+1)e^{-t}$.

- h. Why is the solution unique? Because the given differential equation is normal at $t = 0$. **Or:** Because the existence- and-uniqueness theorem applies since $f(t, x) = -\frac{t}{t+1}x$ is continuous at $(0, 2)$ and so is its x -derivative.

2. (5 points)

- a. Determine the largest rectangular region of the t - x plane that contains the point (t_0, a) and on which the hypotheses of the existence and uniqueness theorem hold for the given o.d.e.

Solution: $0 < t < \pi, -\infty < x < \infty$.

- b. By solving the o.d.e., find the largest interval of t -values on which the solution of the o.d.e. that satisfies $x(t_0) = \alpha$ is defined and has values in the rectangular region found in (a).

$$\frac{dx}{dt} = \cot t \quad x\left(\frac{\pi}{2}\right) = 0$$

Solution: $x = \int x' dt = \int \cot t dt = \int \frac{\cos t}{\sin t} dt = \ln|\sin t| + C$, and $0 = x(\pi/2) = \ln|\sin \pi/2| + C = C$, so $x(t) = \ln \sin t$ and the interval in question is $(0, \pi)$.

3. (10 points) Use Cramer's rule to solve for x only.

$$x + 2y - 3z = 1$$

$$2x + 2y + 3z = 2$$

$$x - y + z = 0$$

Solution: $x = \frac{\det \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}} = \frac{5 + 2}{5 + 2 + 12} = \frac{7}{19}$.

4. (10 points) The Wronskian of t, t^2, t^3 is 0 when $t = 0$ (check this). Why are t, t^2, t^3 independent?

Solution: $\det \begin{pmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix} = t(12t^2 - 6t^2) - (6t^3 - 2t^3) = 2t^3 \begin{cases} = 0 & \text{for } t = 0 \\ \neq 0 & \text{for } t \neq 0 \end{cases}$.

5. (10 points) Solve

$$(D^3 + 3D^2 + 3D + 1)x = 0$$

$$x(0) = x'(0) = x''(0) = 0$$

Solution: By inspection, $x(t) = 0$ for all t is a solution, and by the theorem on existence and uniqueness, it is the only one.

6. (4 points) Use exponential shift to compute

$$(D^2 - 7D + 5)t^2e^{2t}$$

Solution: $(D^2 - 7D + 5)t^2e^{2t} = e^{2t}((D + 2)^2 - 7(D + 2) + 5)t^2 = e^{2t}(D^2 - 3D - 5)t^2 = e^{2t}(2 - 6t - 5t^2)$.

7. (5 points) Find the annihilator of

$$e^t + \sin 2t - 3$$

Solution: $(D - 1)(D^2 + 4)D$.

8. (10 points) Solve $(4D^2 + 1)(D^2 + 2D + 2)x = 0$

Solution: $c_1 \cos t/2 + c_2 \sin t/2 + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$.

9. (10 points) Solve $(3D^2 + 2D - 1)x = 2 \sin t$

Solution: $(3D^2 + 2D - 1)x = (3D - 1)(D + 1)x = 0$ has the general solution $c_1 E^{t/3} + c_2 e^{-t}$; a simplified guess for a particular solution of $(3D^2 + 2D - 1)x = 2 \sin t$ is $k_1 \cos t + k_2 \sin t$. Plugging this in gives $2 \sin t = 3(-k_1 \cos t - k_2 \sin t) + 2(-k_1 \sin t + k_2 \cos t) - (k_1 \cos t + k_2 \sin t) = (-3k_1 + 2k_2 - k_1) \cos t + (-3k_2 - 2k_1 - k_2) \sin t$, so $-4k_1 + 2k_2 = 0$ and $-2k_1 - 4k_2 = 2$. Cramer's rule gives $k_1 = -1/5$ and $k_2 = -2/5$, so the general solution is $c_1 E^{t/3} + c_2 e^{-t} - (1/5) \cos t - (2/5) \sin t$.

10. (10 points) Solve $(D - 1)x = e^t$

Solution: The associated homogeneous equation has the general solution ce^t . The simplified guess for a particular solution is kte^t ; plugging this into $(D - 1)x = e^t$ gives $e^t = k(D - 1)te^t = ke^t Dt = ke^t$, so $k = 1$ and $x(t) = ce^t + te^t$.

11. (10 points) Use variation of parameters to solve

$$(2D^2 - 4D + 2)x = e^t$$

No credit by any other method.

Solution: In standard form this is $(D^2 - 2D + 1)x = e^t/2$; the associated homogeneous equation has the general solution $c_1 e^t + c_2 t e^t$. Variation of parameters gives

$$\begin{aligned} c_1'(t)e^t + c_2'(t)t &= e^t/2 \\ c_1'(t)e^t + c_2'(t)(t+1)e^t &= e^t/2 \end{aligned}$$

Subtracting the first equation from the second one we find $c_2'(t) = 1/2$, hence $c_2(t) = t/2$, and then $c_1'(t) = -t/2$, so $c_1(t) = -t^2/4$, and $x(t) = c_1 e^t + c_2 t e^t + t^2 e^t/4$.