Mathematics 38 **Differential Equations**
Exam II **October 23, 2006**

1. (10 points)

a. Are the functions $h_1(t) = \begin{cases} 0 & \text{for } t \le 0 \\ t & \text{for } t > 0 \end{cases}$, $h_2(t) = \begin{cases} 0 & \text{for } t \le 0 \\ t^2 & \text{for } t > 0 \end{cases}$ linearly independent on the interval $-\infty < t < \infty$?

Solution: Yes, because $W[h_1, h_2](1) = \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1 \neq 0$.

b. Are they linearly independent on the interval $-\infty < t < 0$?

Solution: No, because $1 \cdot h_1(t) + 1 \cdot h_2(t) = 0$ for $t < 0$.

2. (10 points) What is the Wronskian of $h_1(t) = te^t$ and $h_2(t) = t^2 e^t$ at $t = 0$?

Solution: 0

3. (10 points)

a. What is the general solution of $(D-1)^2(D+1)x = 0$?

Solution: $c_1 e^t + c_2 te^t + c_3 e^{-t}$.

b. What is the general solution of $3(D^2 + D + 2)^2 x = 0$?

Solution: $e^{-t/2}((c_1 + c_2 t)\cos\sqrt{7}t/2 + (c_3 + c_4 t)\sin\sqrt{7}t/2)$.

4. (5 points) Make a *simplified* guess for a particular solution of

$$(D-1)(D^2+1)^3(D+2)x = t^2 e^t + e^{-t}\sin 3t + t.$$

Solution: $c_1 te^t + c_2 t^2 e^t + c_3 t^3 e^t + c_4 e^{-t}\sin 3t + c_5 e^{-t}\cos 3t + c_6 + c_7 t$.

5. (10 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1$ gram, $k = 100$ dynes/cm and $b = 20$ gram/s is undamped, underdamped, critically damped, or overdamped.

Solution: Critically damped: $1 \cdot D^2 + 20 \cdot D + 100 = (D+10)^2$.

6. (10 points) Determine whether the system

$$\begin{aligned} x' &= -ty - z + t \\ y' &= -\frac{x}{t} - \frac{z}{t} + 1 \\ z' &= x - ty \end{aligned}$$

is linear. If it is linear
a. determine whether it is homogeneous, **b.** determine its order, and **c.** write it in matrix form.

Solution: Yes. **a.** no. **b.** 3. **c.**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & -t & -1 \\ -1/t & 0 & -1/t \\ 1 & -t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \text{ or } \vec{x}' = \begin{pmatrix} 0 & -t & -1 \\ -1/t & 0 & -1/t \\ 1 & -t & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}.$$

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7. (10 points) Find the general solution of

$$(D^2 - 2D + 1)x = e^t\sqrt{t} \qquad \text{(for } t > 0).$$

Solution: $c_1e^t + c_2te^t + \frac{4}{15}t^{5/2}e^t$.

8. (15 points) Given the o.d.e.

$$(D^2 - 1)x = t \qquad \text{(N)}$$

- **a.** find the equivalent system $(\mathrm{S_N})$,
- **b.** find the general solution of (N) and use it to obtain the general solution of $(\mathrm{S_N})$,
- **c.** write $(\mathrm{S_N})$ in matrix form,
- **d.** write the general solution of $(\mathrm{S_N})$ in the form $\mathbf{x} = c_1\mathbf{h}_1(t) + \cdots + c_n\mathbf{h}_n(t) + \mathbf{p}(t)$.

Solution: **a.** $\begin{cases} x_1' = x_2 \\ x_2' = x_1 + t \end{cases}$. **b.** $\begin{cases} x_1(t) = c_1e^{-t} + c_2e^t - t \\ x_2(t) = -c_1e^{-t} + c_2e^t - 1 \end{cases}$.

c. $\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}$. **d.** $\mathbf{x} = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix} + \begin{pmatrix} -t \\ -1 \end{pmatrix}$.

9. (10 points) Given the matrix $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$, the vector-valued function $\mathbf{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}$, and the formulas $\begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}$ describing a collection of solutions of the nonhomogeneous system $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$, decide whether the collection is complete.

Solution: Yes, because $\det\begin{pmatrix} 2e^{-2t} & e^{-t} \\ -e^{-2t} & -e^{-t} \end{pmatrix} = -e^{-3t} \neq 0$.

10. (10 points)

a. The vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

are linearly independent. Decide whether $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly independent. Explain clearly.

Solution: If $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$ then also $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + 0\mathbf{v}_5 = 0$. By linear independence of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ this implies $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$ (and $0 = 0$).

b. Show that any set of n-vectors that includes $\mathbf{0}$ is linearly dependent.

Solution: Write the set of vectors as $\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then $5 \cdot \mathbf{0} + 0 \cdot \mathbf{v}_2 + \cdots + 0 \cdot \mathbf{v}_k = 0$, which shows (because $5 \neq 0$) that $\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are dependent.

END OF EXAMINATION