Mathematics 38 **Differential Equations**
Exam II **October 27, 2008**

1. (4 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1$ gram, $k = 5$ dynes/cm and $b = 4$ gram/s is undamped, underdamped, critically damped, or overdamped.

Solution: Underdamped: $1 \cdot r^2 + 4 \cdot r + 5 = (r+2)^2 + 1$ has complex roots.

2. (6 points) Determine whether the system

$$\begin{aligned} x' &= -ty - z + t^2 \\ y' &= -\frac{x}{t} - \frac{z}{t} + 1 \\ z' &= x - ty \\ w' &= tx - y\sqrt{3} + z\sin t + w \end{aligned}$$

is linear. If it is linear

a. determine whether it is homogeneous, **b.** determine its order, and **c.** write it in matrix form.

Solution: Yes. **a.** no. **b.** 4. **c.** $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}' = \begin{pmatrix} 0 & -t & -1 & 0 \\ -1/t & 0 & -1/t & 0 \\ 1 & -t & 0 & 0 \\ t & -\sqrt{3} & \sin t & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} t^2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or

$$\vec{x}' = \begin{pmatrix} 0 & -t & -1 & 0 \\ -1/t & 0 & -1/t & 0 \\ 1 & -t & 0 & 0 \\ t & -\sqrt{3} & \sin t & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} t^2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

3. (8 points) Given the differential equation

$$(D^2 + 4D + 3)x = 3t + 7 \qquad \text{(N)}$$

a. find the equivalent system $(\mathrm{S_N})$,

b. the general solution of (N) is $x(t) = c_1 e^{-t} + c_2 e^{-3t} + t + 1$. *You do not need to verify this.* Use the general solution of (N) to obtain each component of the general solution of $(\mathrm{S_N})$,

c. write $(\mathrm{S_N})$ in matrix form,

d. write the general solution of $(\mathrm{S_N})$ in the form $\vec{x} = c_1 \vec{h}_1(t) + \cdots + c_n \vec{h}_n(t) + \vec{p}(t)$.

Solution: a. $\begin{cases} x_1' = x_2 \\ x_2' = -3x_1 - 4x_2 + 3t + 7 \end{cases}$. **b.** $\begin{cases} x_1(t) = c_1 e^{-t} + c_2 e^{-3t} + t + 1 \\ x_2(t) = -c_1 e^{-t} - 3c_2 e^{-3t} + 1 \end{cases}$.

c. $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 3t+7 \end{pmatrix}$. **d.** $\vec{x} = c_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix} + \begin{pmatrix} t+1 \\ 1 \end{pmatrix}$.

4. (8 points) In parts **a.** and **b.** you are given a matrix A, a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix} : \qquad \begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}.$

Solution: The Wronskian at 0 is $\det\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = -1 \neq 0$, so this is a complete set.

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} : \qquad \begin{cases} x_1 = 6c_1e^{4t} - 2c_2e^{-4t} \\ x_2 = 2c_1e^{4t} - 6c_2e^{-4t} \\ x_3 = \quad c_1e^{4t} + c_2e^{-4t} - 2 \end{cases}.$

Solution: Since this is a linear combination of only 2 (not 3) solutions, this set is not complete.

5. (6 points) Rewrite $f(t) = \begin{cases} 4t+1 & t < 12 \\ 0 & 12 \leq t < 30 \\ t^2 & t \geq 30 \end{cases}$ in unit step function notation.

Solution: $f(t) = 4t + 1 + u_{12}(t)(-(4t+1)) + u_{30}(t)(t^2)$.

6. **a.** (5 points) Compute $\mathscr{L}[e^{1-2t}]$ using the definition. *No credit by any other method*

b. (3 points) State for which values of s this Laplace transform is defined.

Solution: $\displaystyle\int_0^\infty e^{-st}e^{-2t+1}\,dt = \lim_{h\to\infty} e\int_0^h e^{-(s+2)t}\,dt = -\frac{e}{s+2}\lim_{h\to\infty}[e^{-(s+2)h} - 1] = \frac{e}{s+2}$ for $s > -2$.

7. (8 points) Find $\cos 5t * 4$.

Solution: $\displaystyle\cos 5t * 4 = 4 * \cos t = \int_0^t 4\cdot\cos 5u\,du = (4/5)[\sin 5u]_0^t = (4/5)\sin 5t.$

8. (8 points) Find the Laplace transform of $f(t) = 5te^{7t}\sin 2t$.

Solution: $\displaystyle\mathscr{L}[5t\sin 2t] = -5\frac{d}{ds}\frac{2}{s^2+4} = \frac{20s}{(s^2+4)^2}$, so $\displaystyle\mathscr{L}[5te^{7t}\sin 2t] = \frac{20(s-7)}{((s-7)^2+4)^2}.$

9. (16 points) Find the inverse Laplace transform of

a. $\dfrac{s+2}{s^2+4s+5}$.

Solution: $\mathscr{L}^{-1}[\dfrac{s+2}{s^2+4s+5}] = \mathscr{L}^{-1}[\dfrac{s+2}{(s+2)^2+1}] = e^{-2t}\mathscr{L}^{-1}[\dfrac{s}{s^2+1}] = e^{-2t}\cos t$.

b. $\dfrac{5s}{(s^2+25)^2}$.

Solution: $\mathscr{L}^{-1}[\dfrac{5s}{(s^2+25)^2}] = \mathscr{L}^{-1}[\dfrac{s}{(s^2+25)} \cdot \dfrac{5}{(s^2+25)}] = \cos 5t * \sin 5t = \dfrac{t}{2}\sin 5t$.

10. (20 points) Solve using the Laplace transform. *No credit by any other method.*

a. $x'' + 4x' + 4x = t^2 e^{-2t}, \qquad x(0) = x'(0) = 0.$

Solution: $(s+2)^2\mathscr{L}[x] = \mathscr{L}[t^2e^{-2t}] = \dfrac{2}{(s+2)^3}$, so $\mathscr{L}[x] = \dfrac{2}{(s+2)^5}$ and $x = \dfrac{1}{12}t^4e^{-2t}$.

b. $(D-1)x = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases}, \qquad x(0) = 1.$

Solution: $(s-1)\mathscr{L}[x] - 1 = \mathscr{L}[u_2(t)] = e^{-2s}\mathscr{L}[1] = \dfrac{e^{-2s}}{s}$, so $\mathscr{L}[x] = \dfrac{1}{s-1} + \dfrac{e^{-2s}}{s(s-1)}$. Now do a partial fractions decomposition: setting $\dfrac{1}{s(s-1)} = \dfrac{A}{s} + \dfrac{B}{s-1}$ gives $A(s-1) + Bs = 1$, or $A = -1, B = 1$. So,

$$\begin{aligned} x(t) = \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{e^{-2s}}{s(s-1)}\right] &= e^t + u_2(t)\mathscr{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right](t-2) \\ &= e^t + u_2(t)\left(e^{t-2} - 1\right) = e^t - u_2(t)\left(1 - e^t/e^2\right). \end{aligned}$$

11. (8 points) Check this set of vectors for linear independence: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

Solution: Note that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \vec{0}$, so this triple is linearly dependent.

END OF EXAMINATION