Mathematics 38 **Differential Equations**
Exam II **March 10, 2008**

Tufts UNIVERSITY

1. (5 points) Determine whether the system

$$\begin{aligned} x' &= -ty - z + t \\ y' &= -\frac{x}{t} - \frac{z}{t} + 1 \\ z' &= x - ty \end{aligned}$$

is linear. If it is linear
a. determine whether it is homogeneous, **b.** determine its order, and **c.** write it in matrix form.

Solution: Yes. **a.** no. **b.** 3. **c.**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & -t & -1 \\ -1/t & 0 & -1/t \\ 1 & -t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \vec{x}' = \begin{pmatrix} 0 & -t & -1 \\ -1/t & 0 & -1/t \\ 1 & -t & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}.$$

2. (5 points) Given the o.d.e.

$$(D^2 - 1)x = t \qquad \text{(N)}$$

- **a.** find the equivalent system $(\mathrm{S_N})$,
- **b.** the general solution of (N) is $x(t) = c_1 e^{-t} + c_2 e^t - t$. *You do not need to verify this.* Use the general solution of (N) to obtain each component of the general solution of $(\mathrm{S_N})$,
- **c.** write $(\mathrm{S_N})$ in matrix form,
- **d.** write the general solution of $(\mathrm{S_N})$ in the form $\vec{x} = c_1 \vec{h}_1(t) + \cdots + c_n \vec{h}_n(t) + \vec{p}(t)$.

Solution: a. $\begin{cases} x_1' = x_2 \\ x_2' = x_1 + t \end{cases}$. **b.** $\begin{cases} x_1(t) = c_1 e^{-t} + c_2 e^t - t \\ x_2(t) = -c_1 e^{-t} + c_2 e^t - 1 \end{cases}$.
c. $\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}$. **d.** $\vec{x} = c_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + \begin{pmatrix} -t \\ -1 \end{pmatrix}$.

3. (5 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1$ gram, $k = 4$ dynes/cm and $b = 4$ gram/s is undamped, underdamped, critically damped, or overdamped.

Solution: Critically damped: $1 \cdot D^2 + 4 \cdot D + 4 = (D+2)^2$.

4. (5 points) Rewrite $f(t) = \begin{cases} 4t+1 & t < 2 \\ 9 & 2 \le t < 3 \\ t^2 & t \ge 3 \end{cases}$ in unit step function notation.

Solution: $f(t) = 4t + 1 + u_2(t)(9 - (4t+1)) + u_3(t)(t^2 - 9)$.

5. (10 points)

a. Make a simplified guess for a particular solution of $(D^2 - 2D + 1)x = te^t$.
You do not need to determine the coefficients!

Solution: $k_1 t^2 e^t + k_2 t^3 e^t$.

b. Find the general solution of $(D^2 - 2D + 1)x = e^t \sqrt{t} \quad$ for $t > 0$.

Solution:

$$\begin{aligned} c_1'(t)e^t + c_2'(t)e^t t &= 0 \\ c_1'(t)e^t + c_2'(t)e^t(t+1) &= \sqrt{t}e^t \end{aligned} \quad \text{simplifies to} \quad \begin{aligned} c_1'(t) + c_2'(t)t = 0 \\ c_2'(t) = \sqrt{t}, \end{aligned}$$

which gives $c_1'(t) = -t^{3/2}$ and $c_2'(t) = t^{1/2}$ and the general solution $c_1 e^t + c_2 te^t + \dfrac{4}{15} t^{5/2} e^t$.

6. (10 points)

a. Compute $\mathscr{L}[e^{5t+3}]$ using the definition. *No credit by any other method*

b. State for which values of s this Laplace transform is defined.

Solution: $\displaystyle\int_0^\infty e^{-st} e^{5t+3}\, dt = \lim_{h\to\infty} e^3 \int_0^h e^{-(s-5)t}\, dt = -\frac{e^3}{s-5} \lim_{h\to\infty} [e^{-(s-5)h} - 1] = \frac{e^3}{s-5}$
for $s > 5$.

7. (10 points) Find $1 * \cos 5t$.

Solution: $\displaystyle\int_0^t 1 \cdot \cos 5u\, du = (1/5)[\sin 5u]_0^t = (1/5)\sin 5t$.

8. (10 points) Find the Laplace transform of $f(t) = te^{5t}\sin 3t$.

Solution: $\mathscr{L}[t\sin 3t] = -\dfrac{d}{ds}\dfrac{3}{s^2+9} = \dfrac{6s}{(s^2+9)^2}$, so $\mathscr{L}[te^{5t}\sin 3t] = \dfrac{6(s-5)}{((s-5)^2+9)^2}$.

9. (20 points) Find the inverse Laplace transform of

a. $\dfrac{s+3}{s^2+4s+5}$.

Solution: $\mathscr{L}^{-1}[\dfrac{s+3}{s^2+4s+5}] = \mathscr{L}^{-1}[\dfrac{(s+2)+1}{(s+2)^2+1}] = e^{-2t}\mathscr{L}^{-1}[\dfrac{s+1}{s^2+1}] = e^{-2t}(\cos t + \sin t)$.

b. $\dfrac{s}{(s^2+5)^2}$.

Solution: $\mathscr{L}^{-1}[\dfrac{s}{(s^2+5)^2}] = \mathscr{L}^{-1}[\dfrac{s}{(s^2+5)} \cdot \dfrac{1}{(s^2+5)}] = \cos\sqrt{5}t * \dfrac{1}{\sqrt{5}}\sin\sqrt{5}t = \dfrac{t}{2\sqrt{5}}\sin\sqrt{5}t$.

10. (20 points) Solve using the Laplace transform. *No credit by any other method.*

a. $x'' + 4x' + 4x = t^2 e^{-2t}, \qquad x(0) = x'(0) = 0.$

Solution: $(s+2)^2\mathscr{L}[x] = \mathscr{L}[t^2e^{-2t}] = \dfrac{2}{(s+2)^3}$, so $\mathscr{L}[x] = \dfrac{2}{(s+2)^5}$ and $x = \dfrac{1}{12}t^4e^{-2t}$.

b. $(D-1)x = \begin{cases} t^2 & t < 2 \\ t^2+1 & t \geq 2 \end{cases}, \qquad x(0) = 1.$

Solution: $(s-1)\mathscr{L}[x] - 1 = \mathscr{L}[t^2] + \mathscr{L}[u_2(t)] = \dfrac{2}{s^3} + e^{-2s}\mathscr{L}[1] = \dfrac{2}{s^3} + \dfrac{e^{-2s}}{s}$, so

$$\mathscr{L}[x] = \frac{1}{s-1} + \frac{2}{s^3(s-1)} + \frac{e^{-2s}}{s(s-1)}.$$

Now do partial fractions decompositions: setting

$$\frac{1}{s^3(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{E}{s-1}$$

gives

$$As^2(s-1) + Bs(s-1) + C(s-1) + Es^3 = 1 \quad \text{or} \quad A+E = 0,\ B-A = 0, C-B = 0, -C = 1,$$

giving $C = -1$, $B = -1$, $A = -1$, and $E = 1$. Similarly, set

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1},$$

giving $A(s-1) + Bs = 1$, or $A = -1, B = 1$.

So,

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{2}{s^3(s-1)} + \frac{e^{-2s}}{s(s-1)}\right] \\ &= e^t + 2\mathscr{L}^{-1}\left[\frac{-1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s-1}\right] + u_2(t)\mathscr{L}^{-1}\left[\frac{-1}{s} + \frac{1}{s-1}\right](t-2) \\ &= e^t + 2\left(-1 - t - \frac{t^2}{2} + e^t\right) + u_2(t)\left(-1 + e^{t-2}\right) \\ &= 3e^t - 2 - 2t - t^2 - u_2(t)\left(1 - e^{t-2}\right). \end{aligned}$$

END OF EXAMINATION