Mathematics 38
Exam II

Tufts UNIVERSITY

Differential Equations
March 10, 2008

1. (5 points) Determine whether the system

$$\begin{aligned} x' &= -ty - z + t \\ y' &= -\frac{x}{t} - \frac{z}{t} + 1 \\ z' &= x - ty \end{aligned}$$

is linear. If it is linear
a. determine whether it is homogeneous, **b.** determine its order, and **c.** write it in matrix form.

2. (5 points) Given the o.d.e.

$$(D^2 - 1)x = t \qquad \text{(N)}$$

a. find the equivalent system $(\mathrm{S_N})$,
b. the general solution of (N) is $x(t) = c_1 e^{-t} + c_2 e^t - t$. *You do not need to verify this.* Use the general solution of (N) to obtain each component of the general solution of $(\mathrm{S_N})$,
c. write $(\mathrm{S_N})$ in matrix form,
d. write the general solution of $(\mathrm{S_N})$ in the form $\vec{x} = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t) + \vec{p}(t)$.

3. (5 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1$ gram, $k = 4$ dynes/cm and $b = 4$ gram/s is undamped, underdamped, critically damped, or overdamped.

4. (5 points) Rewrite $f(t) = \begin{cases} 4t+1 & t < 2 \\ 9 & 2 \le t < 3 \\ t^2 & t \ge 3 \end{cases}$ in unit step function notation.

5. (10 points)
a. Make a simplified guess for a particular solution of $(D^2 - 2D + 1)x = te^t$. *You do not need to determine the coefficients!*
b. Find the general solution of $(D^2 - 2D + 1)x = e^t\sqrt{t}$ for $t > 0$.

6. (10 points)
a. Compute $\mathscr{L}[e^{5t+3}]$ using the definition. *No credit by any other method*
b. State for which values of s this Laplace transform is defined.

7. (10 points) Find $1 * \cos 5t$.

8. (10 points) Find the Laplace transform of $f(t) = te^{5t}\sin 3t$.

9. (20 points) Find the inverse Laplace transform of
a. $\dfrac{s+3}{s^2+4s+5}$.
b. $\dfrac{s}{(s^2+5)^2}$.

Examination continues on next page

10. (20 points) Solve using the Laplace transform. *No credit by any other method.*

a. $x'' + 4x' + 4x = t^2 e^{-2t}, \qquad x(0) = x'(0) = 0.$

b. $(D-1)x = \begin{cases} t^2 & t < 2 \\ t^2 + 1 & t \geq 2 \end{cases}, \qquad x(0) = 1.$ $(s-1)\mathscr{L}[x] - 1 = \mathscr{L}[t^2] + \mathscr{L}[u_2(t)] = \frac{2}{s^3} + e^{-2s}\mathscr{L}[1] = \frac{2}{s^3} + \frac{e^{-2s}}{s}$, so

$$\mathscr{L}[x] = \frac{1}{s-1} + \frac{2}{s^3(s-1)} + \frac{e^{-2s}}{s(s-1)}.$$

Now do partial fractions decompositions: setting

$$\frac{1}{s^3(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{E}{s-1}$$

gives

$$As^2(s-1) + Bs(s-1) + C(s-1) + Es^3 = 1 \quad \text{or} \quad A + E = 0,\ B - A = 0, C - B = 0, -C = 1,$$

giving $C = -1$, $B = -1$, $A = -1$, and $E = 1$. Similarly, set

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1},$$

giving $A(s-1) + Bs = 1$, or $A = -1, B = 1$.

So,

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{2}{s^3(s-1)} + \frac{e^{-2s}}{s(s-1)}\right] \\ &= e^t + 2\mathscr{L}^{-1}\left[\frac{-1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s-1}\right] + u_2(t)\mathscr{L}^{-1}\left[\frac{-1}{s} + \frac{1}{s-1}\right](t-2) \\ &= e^t + 2\left(-1 - t - \frac{t^2}{2} + e^t\right) + u_2(t)\left(-1 + e^{t-2}\right) \\ &= 3e^t - 2 - 2t - t^2 - u_2(t)\left(1 - e^{t-2}\right). \end{aligned}$$

END OF EXAMINATION