1. (15 points) Solve the differential equation

$$\big((t-1)D^2 - tD + 1\big)x = (t-1)e^t$$

by variation of parameters. You may use that the solution of the associated homogeneous differential equation is $H(t) = c_1 t + c_2 e^t$. [No credit for other methods or if you start with the wrong $H(t)$.]

Solution: Solve

$$\begin{aligned} c_1' t + c_2' e^t &= 0 \\ c_1' + c_2' e^t &= e^t \end{aligned}$$

by Cramer's rule to get

$$c_1' = \frac{\det\begin{pmatrix} 0 & e^t \\ e^t & e^t \end{pmatrix}}{\det\begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix}} = \frac{-e^{2t}}{(t-1)e^t} = -\frac{e^t}{t-1}, \quad c_2' = \frac{\det\begin{pmatrix} t & 0 \\ 1 & e^t \end{pmatrix}}{\det\begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix}} = \frac{te^t}{(t-1)e^t} = 1 + \frac{1}{t-1}.$$

Then $c_1(t) = -\displaystyle\int \frac{e^t}{t-1}\,dt$ and $c_2(t) = \int 1 + \frac{1}{t-1}\,dt = t + \ln|t-1|$.

This gives $x(t) = c_1 t + c_2 e^t - t[\displaystyle\int \frac{e^t}{t-1}\,dt] + e^t[t + \ln|t-1|]$.

2. (10 points) Find $\mathscr{L}[1]$ *using the definition.*

Solution: $\mathscr{L}[1] = \displaystyle\int_0^\infty e^{-st}\,dt = -\lim_{b\to\infty} \frac{1}{s}e^{-st}\Big]_0^b = \frac{1}{s}.$

3. (10 points) Find the following convolutions *using the definition*

a. $e^{2t} * e^{2t}$,

Solution: $e^{2t} * e^{2t} = \displaystyle\int_0^t e^{2(t-u)}e^{2u}\,du = e^{2t}\int_0^t du = te^{2t}.$

b. $e^{2t} * e^{3t}$.

Solution: $e^{2t} * e^{3t} = \displaystyle\int_0^t e^{2(t-u)}e^{3u}\,du = e^{2t}\int_0^t e^u\,du = e^{2t}(e^t - 1) = e^{3t} - e^{2t}.$

4. (15 points) Find the Laplace transform of

a. $\dfrac{d^3}{dt^3}t^4e^{2t}$,

Solution: $\mathscr{L}[\dfrac{d^3}{dt^3}t^4e^{2t}] = s^3\mathscr{L}[t^4e^{2t}] = s^3\dfrac{4!}{(s-2)^5}.$

b. $\begin{cases} 0 & t < \pi \\ \cos\frac{t}{2} & t \geq \pi \end{cases}$,

Solution: $\mathscr{L}[u_\pi(t)\cos\dfrac{t}{2}] = e^{-\pi s}\mathscr{L}[\cos\dfrac{(t+\pi)}{2}] = -e^{-\pi s}\mathscr{L}[\sin\dfrac{t}{2}] = -e^{-\pi s}\dfrac{1/2}{s^2+1/4}.$

c. $t^2e^{5t}\sin 3t$.

Solution: First compute

$$\mathscr{L}[t^2 \sin 3t] = \frac{d^2}{ds^2}\mathscr{L}[\sin 3t] = \frac{d^2}{ds^2}\frac{3}{s^2+9} = -3\frac{d}{ds}\frac{2s}{(s^2+9)^2} = \frac{-6(s^2+9)-24s^2}{(s^2+9)^3} = \frac{18s^2-54}{(s^2+9)^3}.$$

Then $\mathscr{L}[t^2 e^{5t} \sin 3t] = \dfrac{18(s-5)^2-54}{((s-5)^2+9)^3}$

5. (15 points) Find the inverse Laplace transform of

a. $\dfrac{s+5}{s^2-2s+5}$,

Solution: $\mathscr{L}^{-1}[\dfrac{s+5}{s^2-2s+5}] = \mathscr{L}^{-1}[\dfrac{(s-1)+6}{(s-1)^2+4}] = e^t \cos 2t + 3e^t \sin 2t.$

b. $\dfrac{e^{-s}}{s(s+3)}$,

Solution: $\mathscr{L}^{-1}[\dfrac{1}{s(s+3)}] = \dfrac{1}{3}\mathscr{L}^{-1}[\dfrac{(s+3)-s}{s(s+3)}] = \dfrac{1}{3}\mathscr{L}^{-1}[\dfrac{1}{s} - \dfrac{1}{s+3}] = \dfrac{1}{3}[1-e^{-3t}]$, so $\mathscr{L}^{-1}[\dfrac{e^{-s}}{s(s+3)}] = \dfrac{1}{3}u_1(t)[1-e^{-3(t-1)}].$

c. $\dfrac{5}{(s^2+5)^2}$.

Solution:
$\mathscr{L}^{-1}[\dfrac{5}{(s^2+5)^2}] = \mathscr{L}^{-1}[\dfrac{\sqrt{5}}{s^2+5} \cdot \dfrac{\sqrt{5}}{s^2+5}] = \sin\sqrt{5}t * \sin\sqrt{5}t = \dfrac{1}{2\sqrt{5}}\sin\sqrt{5}t - \dfrac{t}{2}\cos\sqrt{5}t.$

6. (15 points) Solve the initial-value problem

$$(D^2+3D+2)x = \begin{cases} e^{2t} & t<1 \\ 0 & t \ge 1 \end{cases}, \qquad x(0)=x'(0)=0$$

using the Laplace transform. No credit by any other method.

Solution:

$$(s^2+3s+2)\mathscr{L}[x] = \mathscr{L}[(1-u_1(t))e^{2t}] = \frac{1}{s-2} - e^2\frac{e^{-s}}{s-2},$$

so $\mathscr{L}[x] = \dfrac{1}{(s+2)(s+1)(s-2)} - e^{2-s}\dfrac{1}{(s+2)(s+1)(s-2)}$. Now,

$$\mathscr{L}^{-1}[\frac{1}{(s+2)(s+1)(s-2)}] = \mathscr{L}^{-1}[\frac{1/4}{s+2} - \frac{1/3}{s+1} + \frac{1/12}{s-2}] = \frac{1}{4}e^{-2t} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{2t},$$

so

$$e^2\mathscr{L}^{-1}[e^{-s}\frac{1}{(s+2)(s+1)(s-2)}] = e^2u_1(t)\cdot[\frac{1}{4}e^{2-2t} - \frac{1}{3}e^{1-t} + \frac{1}{12}e^{2t-2}].$$

Therefore $x(t) = \dfrac{1}{4}e^{-2t} - \dfrac{1}{3}e^{-t} + \dfrac{1}{12}e^{2t} - u_1(t)\cdot[\dfrac{e^4}{4}e^{-2t} - \dfrac{e^3}{3}e^{-t} + \dfrac{1}{12}e^{2t}]$. Note that, as expected, this does not contain e^{2t} when $t \ge 1$.

7. (10 points) Consider the differential equation

$$(D+1)(D-1)(D-2)x = 2.$$

a. Convert this differential equation to a system.

Solution: Since $(D+1)(D-1)(D-2) = D^3 - 2D^2 - D + 2$ the corresponding system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_1 + x_2 + 2x_3 - 2. \end{aligned}$$

b. Write the system in matrix form.

Solution: $\vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$

c. Find the general solution of the differential equation.

Solution: By inspection, $x = 1$ is a particular solution, so the general solution is $x(t) = c_1 e^{-t} + c_2 e^t + c_3 e^{2t} + 1$.

8. (10 points) Determine whether

$$\begin{pmatrix} 1 \\ 5 \\ 7 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -3 \\ -4 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

are linearly independent.

Solution: No, since

$$2\begin{pmatrix} 1 \\ 5 \\ 7 \\ 0 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -3 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

END OF EXAMINATION