Mathematics 38 — Tufts UNIVERSITY — **Differential Equations**

Exam II — **March 15, 2010**

1. (8 points) Find $\mathscr{L}[t]$ *using the definition.*

Solution: $\mathscr{L}[1] = \int_0^\infty te^{-st}\,dt = -\lim_{b\to\infty} \frac{te^{-st}}{-s}\Big]_0^b + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s^2} = \frac{1}{s}\lim_{b\to\infty}\frac{-e^{-st}}{s}\Big]_0^b =$ $\frac{1}{s^2}$ (which checks with the table!).

2. (14 points) Find the following convolutions *using the definition.*

a. $e^{3t} * e^{3t}$,

Solution: $e^{3t} * e^{3t} = \int_0^t e^{3(t-u)}e^{3u}\,du = e^{3t}\int_0^t du = te^{3t}$.

b. $t * e^{5t}$.

Solution: $t*e^{5t} = \int_0^t (t-u)e^{5u}\,du = \frac{t}{5}(e^{5t}-1) - \frac{1}{5}ue^{5u}\Big]_0^t + \frac{1}{5}\int_0^t e^{5u}\,du = -\frac{t}{5} + \frac{1}{25}(e^{5t}-1)$.

3. (6 points) Find $\mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$ using convolutions.

Solution: $\mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}\right] * \mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 * \cos t = \int_0^t \sin t\,dt =$ $-\cos u\Big]_0^t = 1-\cos t$. (To check this note that $\frac{1}{s(s^2+1)} = \frac{(s^2+1) - s\cdot s}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$.)

4. (21 points) Find the inverse Laplace transform of the following functions.

a. $\frac{s-3}{s^2-4s+6}$.

Solution: $\mathscr{L}^{-1}\left[\frac{s-3}{s^2-4s+6}\right] = \mathscr{L}^{-1}\left[\frac{(s-2)-1}{(s-2)^2+2}\right] = e^{2t}\mathscr{L}^{-1}\left[\frac{s-1}{s^2+2}\right] = e^{2t}\cos\sqrt{2}t -$ $\frac{e^{2t}}{\sqrt{2}}\sin\sqrt{2}t$.

b. $\frac{s^2+5s-12}{(s-3)^2(s+1)}$.

Solution: $\mathscr{L}^{-1}\left[\frac{s^2+5s-12}{(s-3)^2(s+1)}\right] = \mathscr{L}^{-1}\left[\frac{2}{s-3} + \frac{3}{(s-3)^2} - \frac{1}{s+1}\right] = 2e^{3t} + 3te^{3t} - e^{-t}$.

c. $\frac{se^{\pi s}}{s^2+4}$ (you must simplify your answer).

Solution: $\mathscr{L}^{-1}\left[\frac{se^{\pi s}}{s^2+4}\right] = u_{-\pi}(t)\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right](t-(-\pi)) = u_{-\pi}(t)\cos 2(t+\pi) = u_{-\pi}(t)\cos 2t$.

5. (10 points) Solve $(D^2+1)x = \begin{cases} \sin 2t & t<\pi \\ -\sin 2t & t \ge \pi \end{cases}$, $\quad x(0) = x'(0) = 0$.

Solution: We will use that $\sin(t-\pi) = -\sin t$ and $\sin(x+2\pi) = \sin x$.

$$(s^2+1)\mathscr{L}[x] = \mathscr{L}[(D^2+1)x] = \mathscr{L}[\sin 2t - 2u_\pi(t)\sin(2(t+\pi))] = \frac{2}{s^2+4} - 2e^{-\pi s}\frac{2}{s^2+4}.$$

Since

$$\mathscr{L}^{-1}[\frac{1}{(s^2+1)(s^2+4)}] = \frac{1}{3}\mathscr{L}^{-1}[\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)}] = \frac{1}{3}(\sin t - \frac{1}{2}\sin 2t),$$

we get $x(t) =$

$$\mathscr{L}^{-1}[\frac{2}{(s^2+1)(s^2+4)} - 2e^{-\pi s}\frac{2}{(s^2+1)(s^2+4)}] = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t + \frac{2}{3}u_\pi(t)(2\sin t + \sin 2t).$$

6. (15 points) Consider the differential equation

(N) $$D(D+1)(D-1)x = t.$$

The corresponding homogeneous equation $D(D+1)(D-1)x = 0$ has the general solution $H(t) = c_1 + c_2e^{-t} + c_3e^t$.

a. Find the general solution of (N) by whatever method you prefer.

Solution: Variation of parameters would be foolish here. Inspection works. The annihilator method gives the simplified guess $p(t) = At + Bt^2$. Plug into $(D^3 - D)x = t$ to get $p(t) = -t^2/2$ and $x(t) = c_1 + c_2e^{-t} + c_3e^t - t^2/2$.

b. Write (N) as a 3×3 nonhomogeneous system.

Solution: As usual,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_2 + t \end{aligned}$$

or

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}.$$

c. Use **a.** to find the general solution of the system in **b.**.

Solution: As usual,

$$\begin{aligned} x_1(t) = x(t) &= c_1 + c_2e^{-t} + c_3e^t - t^2/2 \\ x_2(t) = x'(t) &= \quad - c_2e^{-t} + c_3e^t - t \\ x_3(t) = x''(t) &= \quad\quad c_2e^{-t} + c_3e^t - 1 \end{aligned}$$

7. (21 points) Find the Laplace transform of the following functions.

a. $te^{3t}\cos 7t$.

Solution: $\mathscr{L}[te^{3t}\cos 7t](s) = \mathscr{L}[t\cos 7t](s-3) = \dfrac{(s-3)^2 - 49}{((s-3)^2 + 49)^2}$

because $\dfrac{d}{ds}\dfrac{s}{s^2+49} = \dfrac{s^2 + 49 - s\cdot 2s}{(s^2+49)^2} = \dfrac{-s^2+49}{(s^2+49)^2}$.

b. $\begin{cases} 1 & 0 \le t < 4 \\ 0 & 4 \le t < 8 \\ e^{2t} & 8 \le t \end{cases}.$

Solution: $\mathscr{L}[1 - u_4(t) + u_8(t)e^{2t}] = \dfrac{1}{s} - \dfrac{e^{-4s}}{s} + e^{-8s}\mathscr{L}[e^{2(t+8)}] = \dfrac{1}{s} - \dfrac{e^{-4s}}{s} + \dfrac{e^{-8(s-2)}}{s-2}$.

c. $\cos^2 t$. (Hint: Use a double-angle formula.)

Solution: $\mathscr{L}[\cos^2 t] = \mathscr{L}[\frac{1}{2} + \frac{1}{2}\cos 2t] = \dfrac{1}{2s} + \dfrac{1}{2}\dfrac{s}{s^2+4}$.

8. (5 points) Compute $\mathscr{L}[\sin^2 t] + \mathscr{L}[\cos^2 t]$.

Solution: $\mathscr{L}[\sin^2 t] + \mathscr{L}[\cos^2 t] = \mathscr{L}[\sin^2 t + \cos^2 t] = \mathscr{L}[1] = \dfrac{1}{s}$.

END OF EXAMINATION

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