Mathematics 38 **Tufts UNIVERSITY** **Differential Equations**
Examination 2 **November 1, 2010 12:00 - 1:20**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (10 points) Use the definition of the Laplace transform to find

$$\mathscr{L}[t \cdot e^{3t}]$$

No credit by any other method.

Solution: $\int_0^\infty te^{3t}e^{-st}\,dt = \int_0^\infty te^{(3-s)t}\,dt = \left[\frac{te^{(s-3)t}}{3-s}\right]_{t=0}^{t\to\infty} - \frac{1}{3-s}\int_0^\infty e^{(3-s)t}\,dt = 0 - \frac{1}{3-s}\left[e^{(3-s)t}\right]_{t=0}^{t\to\infty}$ $= \frac{1}{(3-s)^2} = \frac{1}{(s-3)^2}$ (which agrees with what one finds from the table and the First Shift Formula).

2. (15 points) Find the Laplace transform of the following functions:

a. $e^{-t}\cos 3t$

Solution: $\mathscr{L}[e^{-t}\cos 3t](s) = \frac{s+1}{(s+1)^2+9}$ (First Shift Formula).

b. $te^{2t}\sin 2t$

Solution: $\mathscr{L}[t\sin 2t](s) = -\frac{d}{ds}\frac{2}{s^2+4} = \frac{4s}{(s^2+4)^2}$ (Second Differentiation Formula), so $\mathscr{L}[te^{2t}\sin 2t](s) = \frac{4(s-2)}{([s-2]^2+4)}$ (First Shift Formula).

c. $\begin{cases} 0 & t < 1 \\ t & 1 \le t < 2 \\ 2(t-1)^2 & 2 \le t \end{cases}$

Solution: $\mathscr{L}\big[u_1(t)\cdot t + u_2(t)\cdot[2(t-1)^2 - t]\big](s) = e^{-s}\mathscr{L}[t+1](s) + e^{-2s}\mathscr{L}[2(t+1)^2 - (t+2)](s) =$ $e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right] + e^{-2s}\mathscr{L}[2t^2+3t](s) = e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right] + e^{-2s}\left[\frac{4}{s^3}+\frac{3}{s^2}\right]$ (Second Shift Formula).

3. (15 points) Find the inverse Laplace transform of the following functions:

a. $\dfrac{e^{-4s}}{s+9}$

Solution: $\mathscr{L}^{-1}\left[\dfrac{e^{-4s}}{s+9}\right](t) = u_4(t)\mathscr{L}\left[\dfrac{1}{s+9}\right](t-4) = u_4(t)e^{-9(t-4)}$ (Second Shift Formula).

b. $\dfrac{2s-1}{s^2+4s+5}$

Solution:

$$\mathscr{L}^{-1}\left[\frac{2s-1}{s^2+4s+5}\right](t) = \mathscr{L}^{-1}\left[\frac{2(s+2)-5}{(s+2)^2+1}\right](t) = e^{-2t}(2\cos t - 5\sin t) \quad \text{(First Shift Formula).}$$

c. $\dfrac{1}{s^2(s^2-4)}$

Solution: $\mathscr{L}^{-1}\left[\dfrac{1}{s^2(s^2-4)}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{4}\dfrac{s^2-(s^2-4)}{s^2(s^2-4)}\right](t) = \dfrac{1}{4}\mathscr{L}^{-1}\left[\dfrac{1}{s^2-4} - \dfrac{1}{s^2}\right](t) =$
$\dfrac{1}{4}\mathscr{L}^{-1}\left[\dfrac{1}{4}\dfrac{(s+2)-(s-2)}{s^2-4} - \dfrac{1}{s^2}\right](t) = \dfrac{1}{4}\mathscr{L}^{-1}\left[\dfrac{1}{4}[\dfrac{1}{s-2} - \dfrac{1}{s+2}] - \dfrac{1}{s^2}\right](t) = \dfrac{1}{16}e^{2t} - \dfrac{1}{16}e^{-2t} - \dfrac{t}{4}.$

4. (10 points) Use the definition of convolution to find

$$t * e^t$$

No credit by any other method

Solution: $t*e^t = e^t * t = \displaystyle\int_0^t e^{t-u}u\,du = e^t\int_0^t e^{-u}u\,du = e^t\left[[-ue^u]_0^t + \int_0^t e^{-u}\,du\right] = e^t\left[-te^{-t} - [e^{-t}-1]\right] =$
$e^t - 1 - t.$

Check: $\mathscr{L}[t*e^t] = \mathscr{L}[t]\cdot\mathscr{L}[e^t] = \dfrac{1}{s^2(s-2)} = \dfrac{s-(s-1)}{s^2(s-1)} = \dfrac{1}{s(s-1)} - \dfrac{1}{s^2} = \dfrac{s-(s-1)}{s(s-1)} - \dfrac{1}{s^2} = \dfrac{1}{s-1} - \dfrac{1}{s} - \dfrac{1}{s^2}$,
so $t*e^t = \mathscr{L}^{-1}[\dfrac{1}{s-1} - \dfrac{1}{s} - \dfrac{1}{s^2}] = e^t - 1 - t$.

5. (10 points) Compute $\mathscr{L}^{-1}\left[\dfrac{1}{s(s+4)}\right]$ using convolution. No credit by any other method.

Solution: $\mathscr{L}^{-1}\left[\dfrac{1}{s(s+4)}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s}\right](t) * \mathscr{L}^{-1}\left[\dfrac{1}{s+4}\right](t) = 1 * e^{-4t} = \displaystyle\int_0^t e^{-4u}\,du = \left[\dfrac{e^{-4u}}{-4}\right]_0^t =$
$\dfrac{1-e^{-4t}}{4}.$

Check: $\mathscr{L}^{-1}\left[\dfrac{1}{s(s+4)}\right](t) = \dfrac{1}{4}\mathscr{L}^{-1}\left[\dfrac{(s+4)-s}{s(s+4)}\right](t) = \dfrac{1}{4}[1-e^{-4t}]$

6. (20 points) Solve:

a. $(D^2+2D+2)x = \begin{cases} 0 & t<\pi \\ e^{-t}\sin t & \pi \le t \end{cases} \qquad x(0)=x'(0)=0$

Solution: By inspection $x(t)=0$ for $t<\pi$. The annihilator method leads us to expect $e^{-t}\cos t$, $e^{-t}\sin t$, $te^{-t}\cos t$, and $te^{-t}\sin t$ for $t\ge\pi$. $((s+1)^2+1)\mathscr{L}[x](s) = \mathscr{L}[u_\pi(t)e^{-t}\sin t](s) = e^{-\pi s}\mathscr{L}[e^{-(t+\pi)}\sin(t+\pi)](s) = -e^{-\pi s}e^{-\pi}\mathscr{L}[e^{-t}\sin t](s) = -e^{-\pi s}e^{-\pi}\frac{1}{(s+1)^2+1}$ (Second Shift Formula).

The side calculation $\mathscr{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right](t) = \sin t * \sin t = \frac{1}{2}\sin t - \frac{t}{2}\cos t$ then gives

$x(t) = \frac{1}{2}u_\pi(t)e^{-t}(\sin t - (t-\pi)\cos t)$.

b. $(D^2+4D+4)x = te^{-2t} \qquad x(0)=0, \quad x'(0)=1$

Solution: The annihilator method leads us to expect e^{-2t}, te^{-2t}, t^2e^{-2t}, t^3e^{-2t}. $(s+2)^2\mathscr{L}[x](s)-1 = \mathscr{L}[te^{-2t}](s) = \frac{1}{(s+2)^2}$ (First Shift Formula), so $x(t) = \mathscr{L}^{-1}\left[\frac{1}{(s+2)^4}\right](t) + \mathscr{L}^{-1}\left[\frac{1}{(s+2)^2}\right](t) = \frac{t^3}{6}e^{-2t} + te^{-2t}$ (First Shift).

7. (10 points) Check for independence

a. $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$

Solution: $c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \vec{0}$ implies $\begin{aligned} c_1+c_2&=0\\ c_2+c_3&=0.\\ c_1+c_2+c_3&=0\end{aligned}$ Subtracting the second equation from the last gives $c_1=0$; subtracting the first from the last gives $c_3=0$, and any of the equations then gives $c_2=0$, so these vectors are linearly independent.

b. $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}, \begin{pmatrix}7\\6\\5\\4\end{pmatrix}$

Solution: $3\cdot\left[\begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \begin{pmatrix}7\\6\\5\\4\end{pmatrix}\right] = 2\cdot\left[\begin{pmatrix}5\\6\\7\\8\end{pmatrix} + \begin{pmatrix}7\\6\\5\\4\end{pmatrix}\right]$, so $3\begin{pmatrix}1\\2\\3\\4\end{pmatrix} - 2\begin{pmatrix}5\\6\\7\\8\end{pmatrix} + \begin{pmatrix}7\\6\\5\\4\end{pmatrix} = \vec{0}$, and these vectors are linearly dependent.

8. (10 points)

a. Find the eigenvalues of the matrix $\begin{pmatrix}1&-1&-1\\0&0&-1\\0&0&2\end{pmatrix}$

Solution: 1, 0, 2 since the matrix is triangular.

b. Find the corresponding eigenvector(s) for each eigenvalue.

Solution: 1 has eigenvector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ by inspection.

For the eigenvalue 0, the last row tells us that the last component of an eigenvector is zero, and the first row then tells us that the first 2 components are equal; $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is therefore an eigenvector.

For the eigenvalue 2 we need to solve $\begin{pmatrix}-1&-1&-1\\0&-2&-1\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \vec{0}$, which gives the eigenvector $\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

END OF EXAMINATION