

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (10 points) Find $\mathcal{L}[\cos^2 t]$.

Solution: This is **not** the same as $(\mathcal{L}[\cos t])^2$! $\mathcal{L}[\cos^2 t] = \mathcal{L}\left[\frac{1}{4}(e^{it} + e^{-it})^2\right] = \mathcal{L}\left[\frac{1}{2} + \frac{1}{4}(e^{2it} + e^{-2it})\right] = \frac{1}{2s} + \frac{1}{4}\left(\frac{1}{s-2i} + \frac{1}{s+2i}\right) = \frac{1}{2s} + \frac{1}{2} \frac{s}{s^2+4}$.

2. (15 points) Solve $(D^2 + 1)x = \begin{cases} 0 & t < 5 \\ e^{7t} & t \geq 5 \end{cases}$, $x(0) = x'(0) = 0$.

Solution: $(s^2 + 1)\mathcal{L}[x] = \mathcal{L}[(D^2 + 1)x] = \mathcal{L}[u_5(t)e^{7t}] = e^{-5s} \mathcal{L}[e^{7(t+5)}] = \frac{e^{35-5s}}{s-7}$, so (partial fractions done by seizing an opportunity)

$$\mathcal{L}[x] = \frac{e^{35-5s}}{(s^2 + 1)(s - 7)} = \frac{e^{35-5s}}{50} \frac{(s^2 + 1) - (s + 7)(s - 7)}{(s^2 + 1)(s - 7)} = \frac{e^{35-5s}}{50} \left(\frac{1}{s - 7} - \frac{s + 7}{s^2 + 1} \right)$$

and

$$x = \frac{e^{35}}{50} u_5(t) \left(e^{7(t-5)} - \sin(t-5) - 7 \cos(t-5) \right) = \frac{1}{50} u_5(t) \left(e^{7t} - e^{35} \sin(t-5) - 7e^{35} \cos(t-5) \right)$$

Note: This could be done with undetermined coefficients if you are clever about it. The essential (but not whole!) idea is to solve the initial-value problem $(D^2 + 1)x = e^{7t}$, $x(5) = x'(5) = 0$.

3. (10 points) Consider the differential equation

$$(N) \quad D(D + 1)(D - 1)x = t.$$

The corresponding homogeneous equation $D(D + 1)(D - 1)x = 0$ has the general solution $H(t) = c_1 + c_2 e^{-t} + c_3 e^t$.

- a. Find the general solution of (N) by whatever method you prefer.

Solution: Variation of parameters would be foolish here. The method of undetermined coefficients gives the simplified guess $p(t) = At + Bt^2$. Plug into $(D^3 - D)x = t$ to get $p(t) = -t^2/2$. This gives $x(t) = c_1 + c_2 e^{-t} + c_3 e^t - t^2/2$.

- b. Write (N) as a 3×3 nonhomogeneous system.

Solution: As usual, $\begin{matrix} x'_1 = x_2 \\ x'_2 = x_3 \\ x'_3 = x_2 + t \end{matrix}$ or $\vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$.

(The column of zeros corresponds to the absence of a D^2 -term.)

4. (10 points) Make a *simplified* guess for a particular solution of

$$(D - 1)^2(D^2 + 1)^3(D + 2)x = t^2 e^{3t} + e^t + e^{-t} \sin 3t + t^4.$$

Solution: The term e^t involves an interaction with the left-hand side; the rest is straightforward: $c_1 e^{3t} + c_2 t e^{3t} + c_3 t^2 e^{3t} + c_4 t^2 e^t + c_5 e^{-t} \sin 3t + c_6 e^{-t} \cos 3t + c_7 + c_8 t + c_9 t^2 + c_{10} t^3 + c_{11} t^4$.

5. (15 points) Find the general solution of

$$(t^2 D^2 + 4tD + 2)x = t^5 \quad (\text{for } t > 0).$$

Hint: To solve the associated homogeneous equation try solutions of the form t^α or $e^{\lambda t}$.

Solution: The hint gives the general solution $c_1 t^{-1} + c_2 t^{-2}$ of the associated homogeneous equation: Plugging in t^α gives

$$0 = (t^2 D^2 + 4tD + 2)t^\alpha = [\alpha(\alpha - 1) + 4\alpha + 2]t^\alpha = (\alpha - 1)(\alpha - 2)t^\alpha.$$

Inspection suggests a multiple of t^5 as a particular solution of the nonhomogeneous equation. Plugging this in (or variation of parameters) leads to the particular solution $p(t) = t^5/42$, so the general solution is $x(t) = c_1/t + c_2/t^2 + t^5/42$.

6. (10 points) Determine whether the spring modeled by $(mD^2 + bD + k)x = 0$ with $m = 1$ gram, $k = 100$ dynes/cm and $b = 20$ gram/s oscillates around its equilibrium.

Solution: No, it is critically damped. **Or:** No, the roots of the characteristic polynomial are real.

7. (5 points) Determine whether the system

$$\begin{aligned} x' &= -ty - z + t^2 \\ y' &= -\frac{x}{t} - \frac{z}{t} + 1 \\ z' &= x - ty \\ w' &= tx - y\sqrt{3} + z \sin t + w \end{aligned}$$

is linear. If it is linear

a. determine whether it is homogeneous, **b.** determine its order, and **c.** write it in matrix form.

Solution: Yes. **a.** no. **b.** 4. **c.** $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}' = \begin{pmatrix} 0 & -t & -1 & 0 \\ -1/t & 0 & -1/t & 0 \\ 1 & -t & 0 & 0 \\ t & -\sqrt{3} & \sin t & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} t^2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or

$$\vec{x}' = \begin{pmatrix} 0 & -t & -1 & 0 \\ -1/t & 0 & -1/t & 0 \\ 1 & -t & 0 & 0 \\ t & -\sqrt{3} & \sin t & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} t^2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

8. (5 points) In parts **a.** and **b.** you are given a matrix A , a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}$: $\begin{cases} x_1 = 2c_1 e^{-2t} + c_2 e^{-t} \\ x_2 = -c_1 e^{-2t} - c_2 e^{-t} + e^{-t} \end{cases}$

Solution: The Wronskian (of the solutions of the associated homogeneous equation included here)

at 0 is $\det \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = -1 \neq 0$, so this is a complete set.

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $\vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$: $\begin{cases} x_1 = 6c_1 e^{4t} - 2c_2 e^{-4t} \\ x_2 = 2c_1 e^{4t} - 6c_2 e^{-4t} \\ x_3 = c_1 e^{4t} + c_2 e^{-4t} - 2 \end{cases}$

Solution: Since this involves a linear combination of only 2 (not 3) solutions of the associated homogeneous system, this collection is not complete.

9. (10 points) Check this set of vectors for linear independence:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Solution: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \vec{0}$, so this triple is linearly dependent.

10. (5 points) Consider the functions t^3 and $|t^3|$ defined on $-\infty < t < \infty$. Are they linearly independent? Explain!

Solution: If $c_1 t^3 + c_2 |t^3| = 0$ for all t , then we can insert $t = 1$ to get $c_1 + c_2 = 0$ and $t = -1$ to get $-c_1 + c_2 = 0$; together these give $c_1 = 0$ and $c_2 = 0$, so these functions are linearly independent.

11. (5 points, *no credit unless every answer is correct*) For each of the following vectors decide whether

it is an eigenvector of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and if so, provide the corresponding eigenvalue. For each part, your answer should be either “NO” or a number; please put all your answers on the inside front blue cover of your examination booklet.

a. $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ **Solution:** 2: Compute $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

b. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ **Solution:** 5: By inspection or a like computation.

c. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **Solution:** 0—we now have 3 distinct eigenvalues in hand, together with corresponding eigenvectors, so for each remaining part we only need to check whether the given vector is a multiple of one of the 3 we already have; the answer is either “No” or the corresponding eigenvalue.

d. $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ **Solution:** 5

e. $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ **Solution:** 0

f. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **Solution:** 2

g. $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **Solution:** No

h. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **Solution:** No

i. $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ **Solution:** No

j. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **Solution:** No

k. $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ **Solution:** No

l. $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ **Solution:** No

m. $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ **Solution:** No

n. $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ **Solution:** No

o. $\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ **Solution:** No