Tufts UNIVERSITY

Mathematics 38 **Differential Equations**
Examination 3 **December 6, 2010 12:00 - 1:20**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (15 points) The 3×3 matrix

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has a (triple) eigenvalue of 1 (you do not have to verify this). Find the general solution of

$$D\vec{x} = A\vec{x}$$

Solution: $(A-I)^2 = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$; $(A-I)^3 = 0$. $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are generalized eigenvectors. Each of these gives a solution $h_i(t) = e^t[I + t(A-I) + \frac{t^2}{2}(A-I)^2]\vec{v}_i =$

$$e^t \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix} \right] \vec{v}_i = e^t \begin{pmatrix} 1+2t+t^2/2 & -3t-t^2 & t+t^2/2 \\ t+t^2/2 & 1-t-t^2 & t^2/2 \\ t^2/2 & t-t^2 & 1-t+t^2/2 \end{pmatrix} \vec{v}_i,$$

so the sought general solution is

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1+2t+t^2/2 \\ t+t^2/2 \\ t^2/2 \end{pmatrix} + c_2 e^t \begin{pmatrix} -3t-t^2 \\ 1-t-t^2 \\ t-t^2 \end{pmatrix} + c_3 e^t \begin{pmatrix} t+t^2/2 \\ t^2/2 \\ 1-t+t^2/2 \end{pmatrix}$$

2. (10 points) Find all solutions of

$$\begin{pmatrix} 1 & 2 & 1 & -1 & -1 \\ 2 & 2 & 2 & -3 & -2 \\ -1 & 0 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

or explain why the system has no solutions.

Solution:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 1 \\ 2 & 2 & 2 & -3 & -2 & 1 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + R_2 - R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 1 \\ 0 & -2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In terms of the free variables $x_3 =: a$, $x_4 =: b$, $x_5 =: c$ we then find $x_2 = \frac{1}{2}(1-b)$ and $x_1 = -a + 2b + c$.

Or: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Or simply: $\begin{pmatrix} -x_3 + 2x_4 + x_5 \\ \frac{1}{2}(1-x_4) \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

3. (10 points) Solve $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Solution: Note the block form. This is a decoupled pair of second-order systems of differential equations. The characteristic polynomial of $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ is $(1-\lambda)(-1-\lambda)-3 = \lambda^2 - 4$, so the eigenvalues are ± 2. 2 has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by inspection, for $\lambda = -2$ solve $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \vec{v} = 0$ to find $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Next, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$, and solving the second equation of $(A - iI)\vec{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \vec{v} = \vec{0}$ gives eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ with associated solution

$$e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos t + i \sin t) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i\cos t + \sin t \\ \cos t + i \sin t \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

Thus, the general solution is $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ -\cos t \\ \sin t \end{pmatrix}$.

4. (15 points) Solve

$$D\vec{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \vec{x}$$

Solution: The characteristic polynomial is $[(1-\lambda)^2 + 1]^2$, which gives double eigenvalues $1 \pm i$.

$$[A - (1+i)I]^2 = \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{pmatrix}^2 = \begin{pmatrix} -2 & -2i & 1 & 0 \\ 2i & -2 & -2i & -1 \\ 0 & 0 & -2 & 2i \\ 0 & 0 & -2i & -2 \end{pmatrix} \to \begin{pmatrix} -2i & 2 & i & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & -2i & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

gives $\vec{v}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}$ (by inspection a proper eigenvector!) and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2i \end{pmatrix}$.

$\vec{v}_1$ gives rise to the solution

$$e^{(1+i)t}\vec{v}_1 = e^t(\cos t + i \sin t) \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = e^t \begin{pmatrix} \cos t + i \sin t \\ i\cos t - \sin t \\ 0 \\ 0 \end{pmatrix},$$

and $\vec{v}_2$ yields $e^{(1+i)t}\left[\vec{v}_2 + t(A - (1+i)I)\vec{v}_2\right] =$

$$e^t(\cos t + i \sin t) \left[\begin{pmatrix} 1 \\ 0 \\ 2 \\ -2i \end{pmatrix} + t \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2i \end{pmatrix} \right] = e^t \begin{pmatrix} \cos t + i \sin t - it\cos t + t \sin t \\ t\cos t + it \sin t \\ 2\cos t + 2i \sin t \\ -2i\cos t + 2\sin t \end{pmatrix}.$$

Taking real and imaginary parts gives

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} \cos t + t\sin t \\ t\cos t \\ 2\cos t \\ 2\sin t \end{pmatrix} + c_4 e^t \begin{pmatrix} \sin t - t\cos t \\ t \sin t \\ 2\sin t \\ -2\cos t \end{pmatrix}.$$

5. (15 points) Solve

$$D\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solution: The general solution of the associated homogeneous equation is $\vec{h}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (by inspection). Variation of parameters: $\left(\begin{array}{cc|c} e^{2t} & 1 & 0 \\ e^{2t} & -1 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} e^{2t} & 1 & 0 \\ 0 & -2 & 4 \end{array}\right)$ gives $c_2'(t) = -2$ and then $c_1'(t) = 2e^{-2t}$, so $c_2(t) = -2t$, $c_1(t) = -e^{-2t}$, and $\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Initial values: setting $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and reducing (or Cramer's Rule) gives $\vec{x}(t) = \frac{5}{2} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{5}{2} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 + 4t \\ 3 - 4t \end{pmatrix}$.

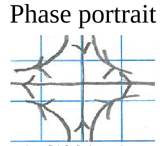
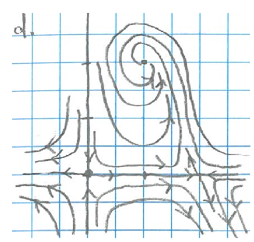
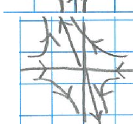
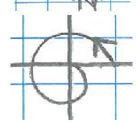
6. (20 points) For the system

$$\frac{dx}{dt} = 4x - 2x^2 - xy$$

$$\frac{dy}{dt} = -y + xy$$

a. Find the equilibria (and determine their stability)
b. Classify each equilibrium as an attractor, a repeller, or neither of these.
c. Draw the phase portrait of the linearization at each equilibrium.
d. Draw the phase portrait of the entire system.

Solution: The linearization matrix is $A_{(x,y)} = \begin{pmatrix} 4 - 4x - y & -x \\ y & x - 1 \end{pmatrix}$.

Equilibrium	$A_{(x,y)}$	Eigenvalues	Stability	Classification	Phase portrait	Total phase portrait
$(0,0)$	$\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$	$4, -1$	unstable	neither		
$(2,0)$	$\begin{pmatrix} -4 & -2 \\ 0 & 1 \end{pmatrix}$	$-4, 1$	unstable	neither		
$(1,2)$	$\begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}$	$-1 \pm i$	stable	attractor		

7. (15 points) For the system

$$\frac{dx}{dt} = x^2 y + y^3$$

$$\frac{dy}{dt} = x^3 + xy^2$$

a. Verify that $G(x,y) = y^2 - x^2$ is a constant of motion.
b. Find the equilibria of the system.
c. Find the critical points of G and classify them as extremum or saddle.
d. Classify each equilibrium of the system as stable or unstable.

Solution: a. $\frac{dG}{dt} = \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} = -2x(x^2 y + y^3) + 2y(x^3 + xy^2) = 0$

b. $0 = x^2 y + y^3 = y(x^2 + y^2)$ and $0 = x^3 + xy^2 = x(x^2 + y^2)$ implies $x = y = 0$, so $(0,0)$ is the only equilibrium.

c. $0 = (\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}) = (-2x, 2y)$ gives $(0,0)$ as the sole critical point, and this is clearly a saddle.

d. Therefore the origin is a saddle, hence an unstable equilibrium (and neither an attractor nor a repeller).

END OF EXAMINATION