Mathematics 38 **Tufts University** **Differential Equations**

Third Examination

Please make sure all electronic devices you carry are turned off, especially calculators, cell phones, and anything that beeps. Pack all these as well as all notes and books away out of sight.

Remember to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on this exam.

You must show all work and cross out anything you do not want graded. Good luck!

1. (15 points) Find the roots of the characteristic polynomial of $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and for each of them find a corresponding eigenvector.

Solution: See Exercise 3.8.3.

2. (10 points) Given the matrix $A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$ and the eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ find

a. the eigenvalue λ of A to which $\vec{v}$ corresponds and

b. the associated solution of $D\vec{x} = A\vec{x}$.

Solution: **a.** $A\vec{v} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\vec{v}$ so $\lambda = -2$. **b.** $e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. (15 points) The matrix $\begin{pmatrix} -1 & 1 & 4 \\ -2 & 2 & 4 \\ -1 & 0 & 4 \end{pmatrix}$ has 2 as a double eigenvalue with generalized eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the associated solution of $D\vec{x} = A\vec{x}$.

Solution: See Exercise 3.9.4.

4. (15 points) Find the general solution of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$. [Hint: $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector.]

Solution: $A\vec{v} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} = (1-i) \begin{pmatrix} 1 \\ i \end{pmatrix} = (1-i)\vec{v}$, so $\lambda = 1 - i$ and the corresponding complex solution is $e^{(1-i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos t - i \sin t) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + ie^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. Thus the general solution is $c_1 e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$.

5. (15 points) Let $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$. The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

Solution: Solve $c_1'(t)e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2'(t)e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ to get $c_1'(t) = 0, c_2'(t) = e^{-t}$, hence $c_1(t) = 0, c_2(t) = -e^{-t}$, so the general solution is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

6. (10 points) Solve

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & -1 & -3 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

if possible. If you think that this is impossible, explain your reasons carefully.

Solution: $\begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & -1 & -3 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2 - R_3} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -3 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{pmatrix}$; the first row implies that $0 = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 + 0 \cdot u_4 = 1$, so there is no solution.

7. (20 points) Find the general solution of $D\vec{x} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \vec{x}$.

Solution: 2 is a triple eigenvalue; 4 is a simple eigenvalue with eigenvector $\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$. Let $M := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} - 2I$, then $M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ and $M^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$, which gives generalized eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, so the general solution is $c_1 e^{2t} \begin{pmatrix} 1 \\ t \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{4t} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.