Mathematics 38 **Differential Equations**
Examination 3 **November 24, 12:00–1:20**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.

Show all work and cross out work you do not want graded!

Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (15 points)

a. The matrix $\begin{pmatrix} 5 & 4 & 3 & 2 \\ 1 & 0 & -1 & -2 \\ 4 & 4 & 4 & 4 \\ -2 & 1 & 2 & 1 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

b. The general solution of $(D-1)(D+1)(D^2+1)x = 0$ is $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4 \sin t$.
Find the solution of this differential equation that satisfies $x(0) = x'(0) = 1$ and $x''(0) = x'''(0) = 2$.

c. The matrix $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}$ has characteristic polynomial $\lambda(\lambda+2)^3$, and

$$(A+2I)^2 = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (A+2I)^3 = \begin{pmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do not need to check this!
Find the general solution of the differential equation $\vec{x}' = A\vec{x}$.

2. (10 points) Let $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$. The general solution of $D\vec{x} = A\vec{x}$ is $c_1e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

3. (10 points) $\begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, e^{2t}\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, e^{-2t}\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$ are solutions of $D\vec{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ -1 & 1 & 0 \end{pmatrix}\vec{x}$.
You do not need to check this!
Prove or disprove that this is a complete set.

Examination continues on next page

4. (20 points) Find the general solution of the systems

a. $D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$.

b. $D\vec{x} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$.

5. (35 points) For each of the (systems of) differential equations below, find the equilibria, determine their stability, and classify each equilibrium as an attractor, a repeller, or neither of these. Draw the phase portrait.

a. $\dfrac{dx}{dt} = x^2 - 1$.

b. $D\vec{x} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \vec{x}$

c. $\begin{cases} \dfrac{dx}{dt} = 2x - 2x^2 - xy \\ \dfrac{dy}{dt} = 4y - 2xy - y^2 \end{cases}$

6. (10 points)

a. Find as many linearly independent eigenvectors as possible that correspond to the eigenvalue -1 of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

b. Find the general solution of $D\vec{x} = A\vec{x}$.
You may use the following information without verifying it: The eigenvalues of A are 0 and -1 and

$$\begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$$

are linearly independent solutions of $D\vec{x} = A\vec{x}$.

END OF EXAMINATION