Mathematics 38 **Differential Equations**
Examination 3 **November 24, 12:00–1:20**

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.
Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (15 points)

a. The matrix $\begin{pmatrix} 5 & 4 & 3 & 2 \\ 1 & 0 & -1 & -2 \\ 4 & 4 & 4 & 4 \\ -2 & 1 & 2 & 1 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

Solution: $\begin{pmatrix} 5 & 4 & 3 & 2 \\ 1 & 0 & -1 & -2 \\ 4 & 4 & 4 & 4 \\ -2 & 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \\ 0 \end{pmatrix} = 8 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, so the eigenvalue is 8.

b. The general solution of $(D-1)(D+1)(D^2+1)x = 0$ is $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$.
Find the solution of this differential equation that satisfies $x(0) = x'(0) = 1$ and $x''(0) = x'''(0) = 2$.

Solution: $x'(t) = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t$, $x''(t) = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$,
$x'''(t) = c_1e^t - c_2e^{-t} + c_3\sin t - c_4\cos t$, so reduce

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & -1 & 2 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1 - \frac{1}{2}R_3 + \frac{1}{2}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right).$$

Thus, $x(t) = (3/2)e^t - (1/2)\cos t - (1/2)\sin t$.

c. The matrix $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}$ has characteristic polynomial $\lambda(\lambda+2)^3$, and

$$(A+2I)^2 = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (A+2I)^3 = \begin{pmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do not need to check this!
Find the general solution of the differential equation $\vec{x}' = A\vec{x}$.

Solution: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for eigenvalue 0. For $\lambda = -2$, $(A+2I)^3$ reduces to $\begin{pmatrix} 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, yielding generalized eigenvectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$. We have $(A+2I) = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{pmatrix}$,
so $(A+2I)\vec{v}_1 = \begin{pmatrix} -2 \\ 2 \\ 4 \\ -4 \end{pmatrix}$, $(A+2I)\vec{v}_2 = \vec{0}$ and $(A+2I)\vec{v}_3 = \begin{pmatrix} -2 \\ 2 \\ 4 \\ -4 \end{pmatrix}$ as well as $(A+2I)^2\vec{v}_i = \vec{0}$ for each i.

This gives the general solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-2t} (\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ 4 \\ -4 \end{pmatrix}) + c_3 e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + c_4 e^{-2t} (\begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ 4 \\ -4 \end{pmatrix}) .$$

2. (10 points) Let $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$. The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

Solution:

$c_1'(t) e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2'(t) e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \Rightarrow c_1'(t) = \dfrac{\det \begin{pmatrix} e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}}{\det \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}} = 0$, $c_2'(t) = \dfrac{\det \begin{pmatrix} 2e^t & e^t \\ e^t & e^t \end{pmatrix}}{\det \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}} = e^{-t}$, hence $c_1(t) = 0$, $c_2(t) = -e^{-t}$, so the general solution is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. (10 points) $\begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$, $e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $e^{2t} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$, $e^{-2t} \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$ are solutions of $D\vec{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ -1 & 1 & 0 \end{pmatrix} \vec{x}$.
You do not need to check this!
Prove or disprove that this is a complete set.

Solution: Taking the middle 3 solutions at $t = 0$ we get $\det \begin{pmatrix} 3 & -1 & -1 \\ 3 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} = 3(1+3) - 3(-1+1) + 1(3+1) = 16 \neq 0$. This shows that these three solutions are linearly independent, and hence they alone form a complete set. Therefore the whole collection is also complete. Alternatively, check directly via row reduction, say, that with these 5 solutions one can match any given initial condition.

4. (20 points) Find the general solution of the systems

a. $D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$.

Solution: Since there is a column of 0s, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector for eigenvalue 0 by inspection. A closer look shows that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is also an eigenvector for the eigenvalue 0, and moreover, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.
This gives the general solution $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
Alternatively, compute $\det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} = -\lambda \cdot \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = -\lambda^2(\lambda - 2)$ and reduce $A - 0 \cdot I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A - 2 \cdot I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

b. $D\vec{x} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$.

Solution: $\det \begin{pmatrix} -\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (2-\lambda)\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = -(\lambda-2)(\lambda^2+1) = 0$ for $\lambda = 2, \pm i$. By inspection, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2. For $\lambda = i$ reduce $\begin{pmatrix} -i & 0 & -1 \\ 0 & 2-i & 0 \\ 1 & 0 & -i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to get the eigenvector $\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$ for the eigenvalue i, which gives the corresponding solution $(\cos t + i \sin t)\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i\cos t - \sin t \\ 0 \\ \cos t + i \sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ 0 \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}$, so the general solution is $c_1 e^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ 0 \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}$.

5. (35 points) For each of the (systems of) differential equations below, find the equilibria, determine their stability, and classify each equilibrium as an attractor, a repeller, or neither of these. Draw the phase portrait.

a. $\dfrac{dx}{dt} = x^2 - 1$.

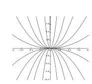
Solution: $x^2 - 1 = (x-1)(x+1) > 0$ when $|x| > 1$, $x^2 - 1 < 0$ on $(-1, 1)$: →−•−←−•→−. -1 is an attractor, hence stable, 1 a repeller, hence unstable.

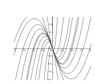
b. $D\vec{x} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \vec{x}$

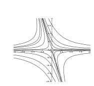
Solution: $(0,0)$ is the sole equilibrium. $\det \begin{pmatrix} 3-\lambda & 2 \\ -2 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 + 4 = 0$ for $\lambda = 3 \pm 2i$, so we have outward spirals, clockwise since $\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ has negative y-component. The origin is a repeller, hence unstable.

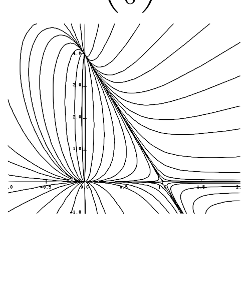
c. $\begin{cases} \dfrac{dx}{dt} = 2x - 2x^2 - xy \\ \dfrac{dy}{dt} = 4y - 2xy - y^2 \end{cases}$

Solution: $\begin{matrix} 0 = 2x - 2x^2 - xy = x(2 - 2x - y) \\ 0 = 4y - 2xy - y^2 = y(4 - 2x - y) \end{matrix}$ gives equilibria $(0,0)$, $(0,4)$, $(1,0)$. The linearizations are

$(0,0)$: $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, $\lambda = 2$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda = 4$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, repeller, unstable 

$(0,4)$: $\begin{pmatrix} -2 & 0 \\ -8 & -4 \end{pmatrix}$, $\lambda = -2$, $\vec{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\lambda = -4$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, attractor, stable 

$(1,0)$: $\begin{pmatrix} -2 & -1 \\ 0 & 2 \end{pmatrix}$, $\lambda = -2$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda = 2$, $\vec{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, saddle, unstable 

The phase portait then is 

6. (10 points)

a. Find as many linearly independent eigenvectors as possible that correspond to the eigenvalue -1 of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Solution: By inspection (or a minimal computation), $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are eigenvectors for the eigenvalue -1.

b. Find the general solution of $D\vec{x} = A\vec{x}$.

You may use the following information without verifying it: The eigenvalues of A are 0 and -1 and

$$\begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$$

are linearly independent solutions of $D\vec{x} = A\vec{x}$.

Solution: The given solutions correspond to the eigenvalue 0 (because their exponential term is $e^{0 \cdot t}$), and there are 3 of them, so we only need to add the two solutions coming from the eigenvalue -1. This gives the general solution

$$c_1 \begin{pmatrix} 1 + (t^2/2) \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix} + c_4 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

END OF EXAMINATION