

No calculators, notes, or books are allowed. Please make sure all electronic devices you carry are turned off and put away out of sight.

Cross out work you do not want graded!

Remember to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (10 points) In parts **a.** and **b.** you are given a matrix A , a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}$: $\begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}$.

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $\vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$: $\begin{cases} x_1 = 6c_1e^{4t} - 2c_2e^{-4t} \\ x_2 = 2c_1e^{4t} - 6c_2e^{-4t} \\ x_3 = c_1e^{4t} + c_2e^{-4t} - 2 \end{cases}$.

2. (10 points) Check the following set of vectors for linear independence: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

3. (5 points) The matrix $\begin{pmatrix} 1 & 4 & 1 & 3 \\ 3 & 7 & -3 & 6 \\ 1 & 3 & 1 & 9 \\ -5 & 8 & 5 & 6 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

4. (15 points)

- a.** Find all solutions, if any, of the system

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 0 \\ -x_1 - x_3 + 2x_4 + x_5 &= 0. \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 - R_2 - R_3} \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array} \right);$$

this is inconsistent and therefore has no solution.

- b.** The general solution of $(D - 1)(D + 1)(D^2 + 1)x = 0$ is $x(t) = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t$. Find the solution of this differential equation that satisfies $x(0) = x'(0) = 1$ and $x''(0) = x'''(0) = 2$.

Examination continues on next page

5. (20 points) Find the general solution of the system

a. $D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}.$

b. $D\vec{x} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}.$

6. (10 points) The matrix $A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ has -1 as an eigenvalue, and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an associated generalized eigenvector. Find the corresponding solution of the differential equation $D\vec{x} = A\vec{x}$.

7. (20 points) Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Hint: The eigenvalues of A are 0 and -1 , and

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A + I)^2 = \begin{pmatrix} 2 & 0 & 0 & -1 & 2 \\ 2 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 & 1 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A + I)^3 = \begin{pmatrix} 4 & 0 & 0 & 1 & 3 \\ 3 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 3 & 0 & 0 & -2 & 3 \\ 3 & 0 & 0 & -3 & 1 \end{pmatrix}.$$

8. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.