Mathematics 38 **Differential Equations**
Examination 3 **April 7, 12:00–1:20**

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1. (10 points) In parts **a.** and **b.** you are given a matrix A, a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix} : \qquad \begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}.$

Solution: The Wronskian at 0 is $\det \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = -1 \neq 0$, so this is a complete set.

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} : \qquad \begin{cases} x_1 = 6c_1e^{4t} - 2c_2e^{-4t} \\ x_2 = 2c_1e^{4t} - 6c_2e^{-4t} \\ x_3 = \ \ c_1e^{4t} + c_2e^{-4t} - 2 \end{cases}.$

Solution: Since this is a generic linear combination of only 2 (not 3) solutions, this set is not complete.

2. (10 points) Check the following set of vectors for linear independence: $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$

Solution: Note that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \vec{0}$, so this triple is linearly dependent. Alternatively, reduce $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \\ 4 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and note the presence of a free variable, which gives the solution -1, 1, 1.

3. (5 points) The matrix $\begin{pmatrix} 1 & 4 & 1 & 3 \\ 3 & 7 & -3 & 6 \\ 1 & 3 & 1 & 9 \\ -5 & 8 & 5 & 6 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

Solution: $\begin{pmatrix} 1 & 4 & 1 & 3 \\ 3 & 7 & -3 & 6 \\ 1 & 3 & 1 & 9 \\ -5 & 8 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, so the eigenvalue is 2.

4. (15 points)

a. Find all solutions, if any, of the system

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 0 \\ -x_1 \qquad - x_3 + 2x_4 + x_5 &= 0. \end{aligned}$$

Solution: $\left(\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2 - R_3} \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array}\right)$;
this is inconsistent and therefore has no solution.

b. The general solution of $(D-1)(D+1)(D^2+1)x = 0$ is $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$.
Find the solution of this differential equation that satisfies $x(0) = x'(0) = 1$ and $x''(0) = x'''(0) = 2$.

Solution: $x'(t) = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t$, $x''(t) = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$, $x'''(t) = c_1e^t - c_2e^{-t} + c_3\sin t - c_4\cos t$, so reduce

$$\begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 1 & -1 & 0 & 1 & | & 1 \\ 1 & 1 & -1 & 0 & | & 2 \\ 1 & -1 & 0 & -1 & | & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 1 & -1 & 0 & 1 & | & 1 \\ 0 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & -2 & | & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \frac{1}{2}R_3 + \frac{1}{2}R_4} \begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 0 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & -2 & | & 1 \end{pmatrix}.$$

Thus, $x(t) = (3/2)e^t - (1/2)\cos t - (1/2)\sin t$.

5. (20 points) Find the general solution of the system

a. $D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\vec{x}$.

Solution: Since there is a column of 0s, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector for eigenvalue 0 by inspection. A closer look shows that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is also an eigenvector for the eigenvalue 0, and moreover, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2. This gives the general solution $c_1e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Alternatively, compute $\det\begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} = -\lambda \cdot \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = -\lambda^2(\lambda-2)$ and reduce $A - 0 \cdot I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A - 2 \cdot I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

b. $D\vec{x} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\vec{x}$.

Solution: $\det\begin{pmatrix} -\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = -(\lambda-2)(\lambda^2+1) = 0$ for $\lambda = 2, \pm i$. By inspection, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue 2. For $\lambda = i$ reduce $\begin{pmatrix} -i & 0 & -1 \\ 0 & 2-i & 0 \\ 1 & 0 & -i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to get the eigenvector $\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$ for the eigenvalue i, which gives the corresponding solution $(\cos t + i\sin t)\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i\cos t - \sin t \\ 0 \\ \cos t + i\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ 0 \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}$, so the general solution is $c_1e^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -\sin t \\ 0 \\ \cos t \end{pmatrix} + c_3\begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}$.

6. (10 points) The matrix $A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ has -1 as an eigenvalue, and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an associated generalized eigenvector. Find the corresponding solution of the differential equation $D\vec{x} = A\vec{x}$.

Solution: $(A+I)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0 & -1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $(A+I)^2\begin{pmatrix}1\\0\\0\end{pmatrix} = (A+I)\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$, so the corresponding solution is $e^{-t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$.

7. (20 points) Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix}0 & 0 & 0 & 0 & 1\\1 & -1 & 0 & -1 & 1\\1 & 0 & -1 & -1 & 0\\0 & 0 & 0 & 0 & 1\\1 & 0 & 0 & -1 & 0\end{pmatrix}.$$

Hint: The eigenvalues of A are 0 and -1, and

$$A^2 = \begin{pmatrix}1 & 0 & 0 & -1 & 0\\0 & 1 & 0 & 0 & -1\\-1 & 0 & 1 & 1 & 0\\1 & 0 & 0 & -1 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix}, \qquad (A+I)^2 = \begin{pmatrix}2 & 0 & 0 & -1 & 2\\2 & 0 & 0 & -2 & 1\\1 & 0 & 0 & -1 & 0\\1 & 0 & 0 & 0 & 2\\2 & 0 & 0 & -2 & 1\end{pmatrix},$$

$$A^3 = \begin{pmatrix}0 & 0 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 1\\1 & 0 & -1 & -1 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix}, \qquad (A+I)^3 = \begin{pmatrix}4 & 0 & 0 & 1 & 3\\3 & 0 & 0 & -1 & 1\\1 & 0 & 0 & -1 & 0\\3 & 0 & 0 & -2 & 3\\3 & 0 & 0 & -3 & 1\end{pmatrix}.$$

Solution: By inspection of A there are (at least) two eigenvectors for the eigenvalue -1, which is therefore at least a double eigenvalue (this can also be seen from the fact that $A+I$ has to zero columns and hence 2 free variables). This implies that the eigenvalue 0 has multiplicity at most 3; but since A^3 has 3 zero rows (and hence 3 free variables), the multiplicity of 0 is 3. *Note: One does not need to know the multiplicities!*

Generalized eigenvectors: In A^3 only the first 2 columns have corner entries, so we get get the generalized eigenvectors $\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}$. For the eigenvalue -1 clearly $\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$ are *actual* eigenvectors.

Therefore, the general solution is

$$
\begin{aligned}
& c_1 \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + tA \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2} A^2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) + c_2 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + tA \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2} A^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) \\
& + c_3 \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + tA \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2} A^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) + c_4 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
= & c_1 \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_2 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \\
& + c_3 \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) + c_4 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
= & c_1 \begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ \frac{t^2}{2} \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix} + c_4 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

8. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.

Solution: Denote the vectors by $\vec{0}, \vec{v}_1, \dots, \vec{v}_n$ and note that in

$$1 \cdot \vec{0} + 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0}$$

not all coefficients are zero.

END OF EXAMINATION