Mathematics 38 **Differential Equations**
Examination 3 **November 23, 12:00–1:20**

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (10 points) In parts **a.** and **b.** you are given a matrix A, a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix} : \quad \begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}.$

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} : \quad \begin{cases} x_1 = 6c_1e^{4t} - 2c_2e^{-4t} \\ x_2 = 2c_1e^{4t} - 6c_2e^{-4t} \\ x_3 = \quad c_1e^{4t} + c_2e^{-4t} - 2 \end{cases}.$

2. (10 points) Determine whether $\begin{pmatrix} 1 \\ 5 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ are linearly independent.

3. (10 points) The matrix $\begin{pmatrix} 6 & -4 & 2 & -3 \\ 9 & 1 & 1 & -1 \\ 7 & 6 & 0 & 5 \\ 7 & 6 & -3 & -3 \end{pmatrix}$ has $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

4. (20 points)

a. Find all solutions, if any, of the system

$$\begin{aligned} 2x_1+2x_2+2x_3-3x_4-2x_5 &= 0 \\ -x_1 \qquad\quad - x_3+2x_4+x_5 &= 0 \\ x_1+2x_2+x_3-x_4-x_5 &= 2. \end{aligned}$$

b. The general solution of $(D-1)(D+1)(D^2+1)x = 0$ is $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$. Find the solution of this differential equation that satisfies $x(0) = x'(0) = 1$ and $x''(0) = x'''(0) = 2$.

5. (10 points) Find the general solution of the system $D\vec{x} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \vec{x}$.

6. (10 points) Find the general solution of $D\vec{x} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 9\sin 9t \\ e^{5t} \end{pmatrix}$

7. (20 points) Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

You may use that $A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

8. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.

END OF EXAMINATION