Mathematics 38 | **Differential Equations**
Examination 3 | **November 23, 12:00–1:20**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!

Remember to sign your blue book.

With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (10 points) In parts **a.** and **b.** you are given a matrix A, a vector-valued function $\vec{E}(t)$ and formulas describing a collection of solutions of the nonhomogeneous system $D\vec{x} = A\vec{x} + \vec{E}(t)$. In each case decide whether the collection is complete.

a. $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 2e^{-t} \\ -e^{-t} \end{pmatrix}: \qquad \begin{cases} x_1 = 2c_1e^{-2t} + c_2e^{-t} \\ x_2 = -c_1e^{-2t} - c_2e^{-t} + e^{-t} \end{cases}.$

Solution: The Wronskian at 0 is $\det \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = -1 \neq 0$, so this is a complete set.

b. $A = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \vec{E}(t) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}: \qquad \begin{cases} x_1 = 6c_1e^{4t} - 2c_2e^{-4t} \\ x_2 = 2c_1e^{4t} - 6c_2e^{-4t} \\ x_3 = c_1e^{4t} + c_2e^{-4t} - 2 \end{cases}.$

Solution: Since this is a generic linear combination of only 2 (not 3) solutions, this set is not complete.

2. (10 points) Determine whether $\begin{pmatrix} 1 \\ 5 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ are linearly independent.

Solution:

$$\begin{pmatrix} 1 & -1 & -1 \\ 5 & -3 & 1 \\ 6 & -4 & 2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 5R_1,\ R_3 \to R_3 - 6R_1} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2 - R_4,\ R_3 \to R_3 - R_2,\ R_2 \to R_2 - 2R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

has 3 corner entries, so the vectors are linearly independent.

3. (10 points) The matrix $\begin{pmatrix} 6 & -4 & 2 & -3 \\ 9 & 1 & 1 & -1 \\ 7 & 6 & 0 & 5 \\ 7 & 6 & -3 & -3 \end{pmatrix}$ has $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue.

Solution: $\begin{pmatrix} 6 & -4 & 2 & -3 \\ 9 & 1 & 1 & -1 \\ 7 & 6 & 0 & 5 \\ 7 & 6 & -3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, so the eigenvalue is 3.

4. (20 points)

a. Find all solutions, if any, of the system

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 0 \\ -x_1 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2. \end{aligned}$$

Solution: $\left(\begin{array}{ccccc|c} 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \\ 1 & 2 & 1 & -1 & -1 & 2 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1 - R_2} \left(\begin{array}{ccccc|c} 2 & 2 & 2 & -3 & -2 & 0 \\ -1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right)$;
this is inconsistent and therefore has no solution.

b. The general solution of $(D-1)(D+1)(D^2+1)x=0$ is $x(t)=c_1e^t+c_2e^{-t}+c_3\cos t+c_4\sin t$. Find the solution of this differential equation that satisfies $x(0)=x'(0)=1$ and $x''(0)=x'''(0)=2$.

Solution: $x'(t)=c_1e^t-c_2e^{-t}-c_3\sin t+c_4\cos t$, $x''(t)=c_1e^t+c_2e^{-t}-c_3\cos t-c_4\sin t$, $x'''(t)=c_1e^t-c_2e^{-t}+c_3\sin t-c_4\cos t$, so reduce

$$\begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 1 & -1 & 0 & 1 & | & 1 \\ 1 & 1 & -1 & 0 & | & 2 \\ 1 & -1 & 0 & -1 & | & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 1 & -1 & 0 & 1 & | & 1 \\ 0 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & -2 & | & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \frac{1}{2}R_3 + \frac{1}{2}R_4} \begin{pmatrix} 1 & 1 & 1 & 0 & | & 1 \\ 0 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & -2 & | & 1 \end{pmatrix}.$$

Thus, $x(t)=(3/2)e^t-(1/2)\cos t-(1/2)\sin t$.

5. (10 points) Find the general solution of the system $D\vec{x}=\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}\vec{x}$.

Solution: $\det\begin{pmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & -\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix} = -(\lambda-2)(\lambda^2+4)=0$ for $\lambda = 2, \pm 2i$. By inspection, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for eigenvalue 2. For $\lambda=2i$ reduce $\begin{pmatrix} -2i & 0 & 2 \\ 0 & 2-2i & 0 \\ -2 & 0 & -2i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to get the eigenvector $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$, which gives the corresponding solution $(\cos 2t + i\sin 2t)\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin 2t - i\cos 2t \\ 0 \\ \cos 2t + i\sin 2t \end{pmatrix} = \begin{pmatrix} \sin 2t \\ 0 \\ \cos 2t \end{pmatrix} + i\begin{pmatrix} -\cos 2t \\ 0 \\ \sin 2t \end{pmatrix}$, so the general solution is $c_1e^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} \sin 2t \\ 0 \\ \cos 2t \end{pmatrix} + c_3\begin{pmatrix} -\cos 2t \\ 0 \\ \sin 2t \end{pmatrix}$.

6. (10 points) Find the general solution of $D\vec{x}=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}\vec{x}+\begin{pmatrix} 3 \\ 9\sin 9t \\ e^{5t} \end{pmatrix}$

Solution: The general solution of the associated homogeneous system is $c_1e^{3t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}+c_3e^{5t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Variation of parameters: $\begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}\begin{pmatrix} c_1'(t) \\ c_2'(t) \\ c_3'(t) \end{pmatrix} = \begin{pmatrix} 3 \\ 9\sin 9t \\ e^{5t} \end{pmatrix}$, so $\begin{pmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{pmatrix} = \int \begin{pmatrix} 3e^{-3t} \\ 9\sin 9t \\ 1 \end{pmatrix} dt = \begin{pmatrix} -e^{-3t} \\ -\cos 9t \\ t \end{pmatrix}$, and the general solution is $\vec{x}(t)=(c_1e^{3t}-1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+(c_2-\cos 9t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}+(c_3+t)e^{5t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

7. (20 points) Find the general solution of $D\vec{x}=A\vec{x}$, where

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

You may use that $A^2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

Solution: By inspection of A, $\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$ are eigenvectors for the eigenvalue -1.

Since A has rows of 0s, 0 is an eigenvalue; multiplicity 3 since A^2 has 3 rows of 0s, so we don't need to compute A^3, and $A^2 \to \begin{pmatrix}0&0&0&0&0\\1&0&-1&-1&0\\0&1&1&0&-1\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$ gives generalized eigenvectors $\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}$.

Therefore, the general solution is

$$
\begin{aligned}
&c_1\left(\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}+tA\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}+\frac{t^2}{2}A^2\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}\right)+c_2\left(\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}+tA\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}+\frac{t^2}{2}A^2\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}\right)\\
&+c_3\left(\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}+tA\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}+\frac{t^2}{2}A^2\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}\right)+c_4e^{-t}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+c_5e^{-t}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\\
=&c_1\left(\begin{pmatrix}1\\-1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\right)+c_2\left(\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\right)\\
&+c_3\left(\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\right)+c_4e^{-t}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+c_5e^{-t}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\\
=&c_1\begin{pmatrix}1\\t-1\\1\\0\\t\end{pmatrix}+c_2\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}+c_4e^{-t}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+c_5e^{-t}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}.
\end{aligned}
$$

8. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.

Solution: Denote the vectors by $\vec{0}, \vec{v}_1, \ldots, \vec{v}_n$ and note that in

$$1 \cdot \vec{0} + 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0}$$

not all coefficients are zero.

END OF EXAMINATION