Tufts University

Mathematics 38 **Differential Equations**
Exam III **April 12, 2010**

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!
Please put the answers to problems 1–2 on the blue book cover in the corresponding box, as shown here:

1	Yes
2	w.
3	e.
4	c.
5	a.n.
6	cos(t)
7	high
8	yes
9	
10	
T	

1. (10 points, no partial credit) Consider the system $D\vec{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} \vec{x}$. The general solution is

a. $c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

b. $c_1 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

c. $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

d. $c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

e. None of the above

Solution: c.—check the eigenvalue–eigenvector pairs.

2. (10 points, no partial credit) Are the following 3 vectors linearly independent? $\begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ -4 \\ 4 \end{pmatrix}$

(*Answer "yes" or "no".* **Don't guess! A wrong answer will receive –5 points!**)

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For the remaining problems show all work; an answer without an explanation, even if correct, will receive no credit.
If you solve any part of a problem by inspection, you must make clear why the solution works.

3. (20 points) The general solution of $D\vec{x} = \begin{pmatrix} 5 & -3 & 0 \\ 3 & -5 & 0 \\ 0 & 1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ is

$$\vec{x} = c_1 e^{4t} \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

You do not have to verify this. Find the solution that satisfies $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

Solution: By inspection, $e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ works (it is the solution with $c_1 = c_2 = 0$, $c_3 = 1$, and it clearly has the correct initial value).

4. (20 points) Solve $D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$.

Solution: By inspection, 0 , $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and 2, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are eigenvalue–eigenvector pairs; the solution therefore is $c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

5. (20 points) Solve $\quad D\vec{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \vec{x}.$

Solution: Note the block structure of the matrix: $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (\lambda^2 - 4)(\lambda^2 + 1)$, so the eigenvalues are ± 2, $\pm i$. $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - 2I = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} + 2I = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$ or inspection give solutions $e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $e^{-2t} \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$. Inspection of the lower right corner gives the eigenvalue–eigenvector pair i, $\begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix}$ and associated "solution" $(\cos t + i \sin t) \begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ \cos t \\ -\sin t \end{pmatrix}$. The general solution therefore is $c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ \sin t \\ \cos t \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ \cos t \\ -\sin t \end{pmatrix}$.

6. (20 points) Solve $\quad D\vec{x} = \begin{pmatrix} 2 & -1 & -4 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x}.$

Solution: By inspection, 2, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvalue–eigenvector pair (and 0 is another eigenvalue). The characteristic polynomial is $(2-\lambda)[(2-\lambda)(-2-\lambda)+4] = (2-\lambda)\lambda^2$, so 0 is a double eigenvalue (but with only one linearly independent eigenvector). Reduce $(A - 0I)^2 = A^2 = \begin{pmatrix} 2 & -1 & -4 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}^2 = \begin{pmatrix} 4 & -8 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to get generalized eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{0t} \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 & -1 & -4 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^{0t} \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 & -1 & -4 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2+3t \\ 1+2t \\ t \end{pmatrix} + c_3 \begin{pmatrix} -1-6t \\ -4t \\ 1-2t \end{pmatrix} \qquad \left(\text{or} \quad c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2+3t \\ 1+2t \\ t \end{pmatrix} + (c_3 + 2c_2) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right).$$

END OF EXAMINATION