Mathematics 38 **Differential Equations**
Final Exam **May 9, 2011 8:30 - 10:30 AM**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (30 points, 5 each) **No partial credit.**

a. Check for independence of:

1) the collection of functions $x,\ x\ln x,\ x\ln x^2$;

2) the collection of vectors

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix},\begin{pmatrix}8\\7\\6\\5\end{pmatrix},\begin{pmatrix}13\\8\\3\\-2\end{pmatrix},\begin{pmatrix}12\\10\\4\\-1\end{pmatrix};$$

b. Use the definition of the Laplace transform to compute $\mathscr{L}[te^{-t}]$;

c. Find $(D^2+2D+1)[e^{2t}\sin t]$;

d. Evaluate $e^t * t$;

e. Find all solutions of the equation

$$x^{'}+x=x^2;$$

f. Solve the initial value problem

$$x^{'}+x=x^2, \qquad x(0)=\frac{1}{4}.$$

2. (6 points) Find the general solution of the non-homogeneous equation

$$x^{'}+x\tan t=\sin 2t.$$

3. (10 points) Consider the following initial value problem:

$$x^{'}=\sqrt{|x|},\ x(0)=0.$$

a. Is the existence and uniqueness theorem applicable?

b. If it is not applicable, does the IVP has a solution?

c. If a solution exists, is it unique? Explain.

Examination continues on other side

4. (8 points) Find the inverse Laplace transform of

a. $\dfrac{2s-1}{s^2-4s+8}$;

b. $\dfrac{e^{-s}s}{s^3+4s^2+4s}$.

5. (8 points) Let the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

be given.

a. A has an eigenvalue $\lambda = 1$. Find an eigenvector corresponding to this eigenvalue;

b. The vector $v = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$ is an eigenvector of A. Find the corresponding eigenvalue.

6. (10 points) Use the Laplace transform method to solve

$$(D^3 - D)x = \begin{cases} 1, & if \;\; t < 2, \\ 0, & if \;\; t \geq 2, \end{cases} \qquad\qquad x(0) = x'(0) = x''(0) = 0.$$

No credit for any other method.

7. (10 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 1 & -1 & -4 \\ 2 & -2 & -2 \\ 1 & 0 & -4 \end{pmatrix} \vec{x}.$$

8. (8 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

9. (10 points) Consider the system

$$\frac{dx}{dt} = -y,$$

$$\frac{dy}{dt} = x - 2y(1 + x^2).$$

a. Show that $E = x^2 + y^2$ is a Lyapunov function for this system;
b. Find equilibrium points;
c. Classify each equilibrium;
d. Find the linearized matrix for each equilibrium point;
e. Draw the phase portrait of the linearization of each equilibrium;
f. Determine whether the system has closed integral curves.

END OF EXAMINATION