

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded**. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (30 points, 5 each) **No partial credit.**

a. Check for independence of:

1) the collection of functions $x, x \ln x, x \ln x^2$;

[dependent; $x \ln x^2 = 2x \ln x$, which is 2 times of the second function]

2) the collection of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} 13 \\ 8 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 12 \\ 10 \\ 4 \\ -1 \end{pmatrix};$$

[dependent; 2 times of the second column minus 3 times of the first equals the third column, implying dependence]

b. Use the definition of the Laplace transform to compute $\mathcal{L}[te^{-t}]$;

$$[\mathcal{L}[te^{-t}] = \int_0^{\infty} te^{-t}e^{-st}dt = \int_0^{\infty} te^{-(s+1)t}dt = \lim_{A \rightarrow \infty} \left(-\frac{1}{s+1}te^{-(s+1)t} \Big|_0^A - \frac{1}{(s+1)^2}e^{-(s+1)t} \Big|_0^A \right) = \frac{1}{(s+1)^2}.]$$

c. Find $(D^2 + 2D + 1)[e^{2t} \sin t]$;

$$[(D^2 + 2D + 1)[e^{2t} \sin t] = (D + 1)^2[e^{2t} \sin t] = e^{2t}(D + 3)^2[\sin t] = e^{2t}[(\sin t)'' + 6(\sin t)' + 9 \sin t] = e^{2t}(8 \sin t + 6 \cos t).]$$

d. Evaluate $e^t * t$;

$$[e^t * t = \int_0^t e^{t-u}u du = e^t \left(\int_0^t e^{-u}u du \right) = e^t (-ue^u \Big|_0^t - e^{-u} \Big|_0^t) = e^t(-te^{-t} - e^{-t} + 1) = e^t - t - 1.]$$

e. Find all solutions of the equation

$$x' + x = x^2;$$

[$x = 0$ and $x = 1$ by inspection. Find other possible solutions using separation:

$$\frac{dx}{x(x-1)} = dt,$$

or

$$\int \frac{dx}{x-1} - \int \frac{dx}{x} = \int dt,$$

or

$$\ln \left| \frac{x-1}{x} \right| = t + \ln C,$$

or

$$x(t) = (1 - Ce^t)^{-1}.$$

Notice, $x = 1$ if $C = 0$. Hence, answer is $x = 0$ and $x(t) = (1 - Ce^t)^{-1}$.]

f. Solve the initial value problem

$$x' + x = x^2, \quad x(0) = \frac{1}{4}.$$

[From Part (e) $x(0) = \frac{1}{1-C} = 1/4$ implying $C = 3$.]

2. (6 points) Find the general solution of the non-homogeneous equation

$$x' + x \tan t = \sin 2t.$$

[First, the homogeneous equation has the general solution $x = c \cos t$. Variation of Parameters gives $p(t) = -2(\cos t)^2$. Hence, $x(t) = C \cos t - 2(\cos t)^2$.]

3. (10 points) Consider the following initial value problem:

$$x' = \sqrt{|x|}, \quad x(0) = 0.$$

a. Is the existence and uniqueness theorem applicable?

[No, since $f(t, x) = \sqrt{|x|}$ is not differentiable at $x_0 = 0$;

b. If it is not applicable, does the IVP has a solution?

[Yes, e.g. $x = 0$ by inspection;]

c. If a solution exists, is it unique? Explain.

[No, e.g. $x(t) = t^2/4$ (which can be easily found by separation of variables) is also a solution to the given IVP.]

4. (8 points) Find the inverse Laplace transform of

a. $\frac{2s-1}{s^2-4s+8}$;

$$\left[\frac{2s-1}{s^2-4s+8} = 2 \frac{s-2}{(s-2)^2+4} + \frac{3}{2} \frac{2}{(s-2)^2+4}, \text{ implying } \mathcal{L}^{-1}\left[\frac{2s-1}{s^2-4s+8}\right] = 2e^{2t} \cos 2t + \frac{3}{2}e^{2t} \sin 2t; \right]$$

b. $\frac{e^{-s}s}{s^3+4s^2+4s}$.

$$\left[\mathcal{L}^{-1}\left[\frac{e^{-s}s}{s^3+4s^2+4s}\right] = \mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+2)^2}\right] = u_1(t) \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right](t-1) = u_1(t)e^{2-2t}(t-1). \right]$$

5. (8 points) Let the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

be given.

a. A has an eigenvalue $\lambda = 1$. Find an eigenvector corresponding to this eigenvalue;

$$\left[v = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right]$$

b. The vector $v = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$ is an eigenvector of A . Find the corresponding eigenvalue.

$$[\lambda = 1 - i.]$$

6. (10 points) Use the Laplace transform method to solve

$$(D^3 - D)x = \begin{cases} 1, & \text{if } t < 2, \\ 0, & \text{if } t \geq 2, \end{cases} \quad x(0) = x'(0) = x''(0) = 0.$$

No credit for any other method.

[Rewrite the equation in the form $(D^3 - D)x = 1 - u_2(t)$ and then apply the Laplace transform. This gives

$$\mathcal{L}[x] = \frac{1}{s^2(s^2-1)} - \frac{e^{-2s}}{s^2(s^2-1)}.$$

Invert the first term

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2-1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2}\right] = \frac{1}{2}(e^t - e^{-t}) - t.$$

Use this and Second Shift formula to invert the second term, obtaining

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2(s^2-1)}\right] = u_2(t) \left(\frac{1}{2}(e^{t-2} - e^{-t+2}) - t + 2 \right).$$

Therefore, $x(t) = \frac{1}{2}(e^t - e^{-t}) - t - \frac{u_2(t)}{2}(e^{t-2} - e^{-t+2} - 2t + 4)$.

7. (10 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 1 & -1 & -4 \\ 2 & -2 & -2 \\ 1 & 0 & -4 \end{pmatrix} \vec{x}.$$

[The characteristic polynomial is

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 & -4 \\ 2 & -2-\lambda & -2 \\ 1 & 0 & -4-\lambda \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 1 & -4-\lambda \end{vmatrix} - (2+\lambda) \begin{vmatrix} 1-\lambda & -4 \\ 1 & -4-\lambda \end{vmatrix}$$

$$-2(\lambda+3) - \lambda(\lambda+2)(\lambda+3) = -(\lambda+3)[(\lambda+1)^2 + 1],$$

giving eigenvalues $\lambda = -3$ and $\lambda = -1 \pm i$. Eigenvectors corresponding to $\lambda = -3$ and $\lambda = -1 - i$ respectively are

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Hence, three linearly independent vector-solutions to the given system are

$$h_1(t) = \begin{pmatrix} e^{-3t} \\ 0 \\ e^{-3t} \end{pmatrix}, h_2(t) = e^{-t} \begin{pmatrix} 3 \cos t - \sin t \\ 3 \cos t + \sin t \\ \cos t \end{pmatrix}, h_3(t) = e^{-t} \begin{pmatrix} \cos t + 3 \sin t \\ -\cos t + 3 \sin t \\ \sin t \end{pmatrix}.$$

]

8. (8 points) Solve the system of differential equations

$$D\vec{x} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

[The homogeneous part has the general solution $H(t) = c_1 h_1(t) + c_2 h_2(t)$, where $h_1(t) = \begin{pmatrix} -e^t \\ e^t \end{pmatrix}$, $h_2(t) = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}$.

The variation of parameters method leads to the system

$$-e^t c_1' + e^{5t} c_2' = 2e^{-2t},$$

$$e^t c_1' + e^{5t} c_2' = e^{-2t}.$$

Solving this system we have $c_1' = -\frac{1}{2}e^{-3t}$ and $c_2' = \frac{3}{2}e^{-7t}$. Integrating, $c_1 = \frac{1}{6}e^{-3t}$ and $c_2 = -\frac{3}{14}e^{-7t}$. Hence, a particular solution is $p(t) = -\frac{1}{21} \begin{pmatrix} 8e^{-2t} \\ e^{-2t} \end{pmatrix}$.

Answer: $x(t) = c_1 \begin{pmatrix} -e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} - \frac{1}{21} \begin{pmatrix} 8e^{-2t} \\ e^{-2t} \end{pmatrix}$.]

9. (10 points) Consider the system

$$\frac{dx}{dt} = -y,$$

$$\frac{dy}{dt} = x - 2y(1 + x^2).$$

a. Show that $E = x^2 + y^2$ is a Lyapunov function for this system;

$$[\frac{dE}{dt} = -xy + xy - 2y^2(1 + x^2) \leq 0]$$

b. Find equilibrium points;

[The only equilibrium point is (0,0).]

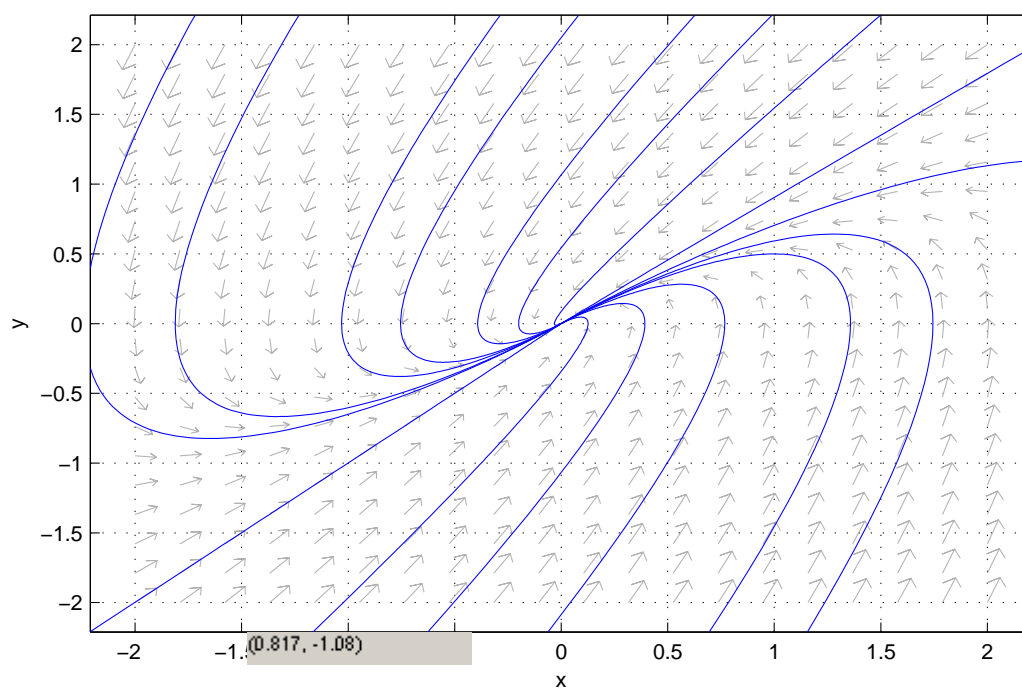
c. Classify each equilibrium;

[$E(x, y) = x^2 + y^2$ has a local (even global) min at (0, 0). Therefore, the equilibrium point is a stable attractor;]

d. Find the linearized matrix for each equilibrium point;

$$[A_{0,0} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}.]$$

e. Draw the phase portrait of the linearization of each equilibrium;



f. Determine whether the system has closed integral curves.
[Existence of the Lyapunov function excludes closed curves.]