Mathematics 38	**Tufts University**	**Differential Equations**

Final Examination

Please make sure all electronic devices you carry are turned off, especially calculators, cell phones, and anything that beeps. Pack all these as well as all notes and books away out of sight.

The exam ends at 10:30. There are wall clocks in front.

Remember to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on this exam. You must show all work and cross out anything you do not want graded. Good luck!

1. (10 points) Find the general solution of $(D-3)^3x = e^{4t}$.

2. (10 points) Find the general solution of $D\vec{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \vec{x}$.

3. (10 points) Solve the initial-value problem $D\vec{x} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

4. (10 points) Find the general solution of $\cos t \dfrac{dx}{dt} + x \sin t = \cos t \sin t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

5. (10 points)

- **a.** Are the functions $h_1(t) = |t|$, $h_2(t) = t$ linearly independent on the interval $-\infty < t < \infty$?
- **b.** The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$ are linearly independent. Decide whether $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are linearly independent. Explain clearly.

6. (5 points) Sketch the phase portrait of the system $D\vec{x} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \vec{x}$.

7. (4 points) Find the equilibria of the system
$$\begin{aligned} \frac{dx}{dt} &= 2xy \\ \frac{dy}{dt} &= 1 + x \end{aligned}.$$

8. (6 points)

- **a.** Verify that $E(x, y) = x^2y - xy^2 + xy$ is a constant of motion for $\begin{aligned} \frac{dx}{dt} &= 2xy - x^2 - x \\ \frac{dy}{dt} &= 2xy - y^2 + y \end{aligned}$.
- **b.** Classify the critical point $(-1/3, 1/3)$ of E as a maximum, a minimum or a saddle point.
- **c.** Classify the equilibrium $(-1/3, 1/3)$ of the system as stable or unstable.

9. (5 points) The system

$$\begin{aligned} \frac{dx}{dt} &= x - x^2 - xy \\ \frac{dy}{dt} &= -y + 2xy \end{aligned}$$

has equilibrium points at $(0,0)$, $(1,0)$ and $(1/2, 1/2)$. Determine the stability of the equilibrium $(1,0)$ and classify it as an attractor, a repeller, or neither of these.

10. (5 points) Compute $\mathscr{L}[e^{7t}]$ **using the definition** of the Laplace transform.

11. (15 points) Compute the following transforms:

- **a.** $\mathscr{L}^{-1}\left[\dfrac{2s+3}{s^2+6s+18}\right]$.
- **b.** $\mathscr{L}[te^{-t}\cos 7t]$.
- **c.** $\mathscr{L}[f(t)]$, where $f(t) = \begin{cases} t^2 & t < 4 \\ e^{6t} & t \geq 4. \end{cases}$

12. (10 points) Solve the initial-value problem $(D^2 - 4)x = \begin{cases} 0 & t < 2 \\ 2 & t \geq 2 \end{cases}, \quad x(0) = 1, \quad x'(0) = 0.$

END OF EXAMINATION