Final Examination

Please make sure all electronic devices you carry are turned off, especially calculators, cell phones, and anything that beeps. Pack all these as well as all notes and books away out of sight. There is a wall clock on the right.

The exam ends at 10:30.

Remember to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on this exam.

You must show all work and cross out anything you do not want graded. Good luck!

1. (3 points each, no partial credit) For each of the differential equations below determine the order, determine whether the differential equation is linear, and if so, whether it is homogeneous.

a. $t^4\dfrac{d^3x}{dt^3}+t\dfrac{dx}{dt}-x-t^7=0$

b. $x^8\dfrac{dx}{dt}+\dfrac{d^7x}{dt^7}=x+t^9$

c. $\left(\dfrac{dx}{dt}\right)^5+\dfrac{d^4x}{dt^4}-t^3x^7+t^7=0$

d. $(x')^2x'''=x^4x''+t^5x'$

2. (3 points each, no partial credit) Find all real values of α for which the given function is a solution of the given differential equation.

a. $x=\alpha,\qquad \dfrac{d^7x}{dt^7}+\dfrac{dx}{dt}-x=7$

b. $x=t^\alpha,\ t>0,\qquad 16t^2xx''+3x^2=0$

c. $x=e^{\alpha t},\qquad x'\sqrt{x}=2e^{3t}$

3. (1 point each, no partial credit) For each of the following differential equations state whether it is normal on $0<t<2$.

a. $(t-1)\dfrac{dx}{dt}-5x=3t$

b. $3\dfrac{dx}{dt}-5x=\csc\pi t$

c. $t\dfrac{dx}{dt}+e^tx=\sin t$

d. $t\sin t\dfrac{dx}{dt}+\pi x=\ln t$

4. (5 points) For each of the vector functions

$$\vec{h}_1(t)=\begin{pmatrix}t^2\\2t\end{pmatrix}\quad\text{and}\quad\vec{h}_2(t)=\begin{pmatrix}t^3\\3t^2\end{pmatrix}$$

determine whether it is a solution of $D\vec{x}=\begin{pmatrix}0&1\\-6/t^2&4/t\end{pmatrix}\vec{x}$.

Examination continues on next page

5. (10 points)

a. Use the method of undetermined coefficients (the "annihilator method") to determine a simplified guess for a particular solution of

$$(D+3)^2(D^2+1)^4 x = \sin t.$$

Obtain the simplest form possible, and leave your answer in terms of the undetermined coefficients.

Do not try to determine the coefficients!

b. Applying the method of undetermined coefficients (the "annihilator method") to determine a simplified guess for a particular solution of

$$(D-2)^2(D-1)x = 2e^{2t}$$

gives the guess $p(t) = kt^2e^{2t}$, where k is a constant. *You do not need to verify this!*

Determine k.

6. (15 points) Find the general solution of $x'' + x = \sec t, \qquad 0 < t < \pi/2.$

7. (15 points) Solve $x'' + 4x' + 4x = te^{-2t}, \qquad x(0) = 0, \quad x'(0) = 1$ using the Laplace transform.
No credit by any other method.

8. (10 points) Find the solution of

$$\begin{aligned} x_1' &= 4x_1 - 3x_2 \qquad & x_1(0) = 1 \\ x_2' &= 5x_1 - 4x_2 \qquad & x_2(0) = 0 \end{aligned}.$$

9. (5 points) Determine whether the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}$$

are linearly independent. Explain!

10. (15 points) Consider the differential equation $\ddot{x} + \cos x = 0$.

a. Convert this differential equation to a system.

b. Find all equilibria of the system.

c. Find the linearization matrix at each equilibrium.

d. Determine for which equilibria the Hartman–Grobman Theorem applies.

e. For the equilibria to which the Hartman–Grobman applies, determine the stability and classify them as attractor, repeller or neither of the two.

f. The differential equation can be interpreted as saying that $F = -\cos x$, where F stands for force (this uses $F = ma$ with $m = 1$). The *potential energy* $U(x)$ satisfies $F = -\dfrac{dU}{dx}$.

Find $U(x)$.

g. Show that the *total energy* $E(x, y) = U(x) + y^2/2$ is a constant of motion.

h. Find the critical points of E and classify each as extremum or saddle point.

i. Determine the stability of each equilibrium of the system of differential equations you obtained in part b.

END OF EXAMINATION