

No calculators, notes, or books are allowed. Please make sure all electronic devices you carry are turned off and put away out of sight.

Remember to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (5 points) Solve $(t^2 + 1)\frac{dx}{dt} = \frac{t}{x}$, $x(1) = 3$.

2. (5 points)

a. Show that A/t is a solution of $tx' + x = 0$.

b. Find the general solution of $t^2x' + tx = 2t$.

3. (10 points) Find the general solution of $(D^2 - D - 2)x = 4t^2 + e^{2t}$.

4. (10 points) Given the differential equation $\frac{dx}{dt} = (x + 2)(x - 2)^2$,

a. find all equilibria,

b. classify each equilibrium as an attractor or a repeller or neither of these,

c. determine the stability of each equilibrium,

d. draw the phase portrait.

5. (10 points) Solve

$$v - 2w + 3x + 2y + z = 10$$

$$2v - 4w + 8x + 3y + 10z = 7$$

$$3v - 6w + 10x + 6y + 5z = 27$$

$$6u - 12w + 21x + 11y + 16z = 44$$

or explain why there is no solution.

6. (5 points) The matrix $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 2 & -3 \end{pmatrix}$ has characteristic polynomial $\lambda(\lambda + 2)^3$,

$$\text{and } (A + 2I)^2 = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A + 2I)^3 = \begin{pmatrix} 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do not need to check this!

Find the general solution of the differential equation $\vec{x}' = A\vec{x}$.

7. (10 points) Let $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$. The general solution of $D\vec{x} = A\vec{x}$ is $c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find the general solution of $D\vec{x} = A\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

8. (5 points) $\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$, $e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $e^{2t} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ are solutions of $D\vec{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ -1 & 1 & 0 \end{pmatrix} \vec{x}$.

You do not need to check this!

Prove or disprove that this is a complete set.

9. (10 points) Consider the pair of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= 1 - x^2 \end{aligned}.$$

- Find all equilibria,
- determine their stability,
- classify each equilibrium as an attractor, a repeller or neither of these.

10. (20 points)

a. Find $\mathcal{L}[te^t \sin t]$.

b. Find $\mathcal{L}^{-1}\left[\frac{s^2 + 1}{s(s + 2)(s - 4)}\right]$.

11. (10 points) Solve $(D^2 + 4)x = 4$, $x(0) = 0$, $x'(0) = 5$ using the Laplace transform.

No credit by any other method.

END OF EXAMINATION