Mathematics 38 | **Differential Equations**
Final Examination | **December 12, 8:30–10:30**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.

Show all work and cross out work you do not want graded!

Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (10 points) For the initial-value problem

$$t\frac{dx}{dt} - x = t^3, \quad x(1) = 0$$

a. State the order of the differential equation, whether the differential equation is linear, homogeneous or nonhomogeneous, and whether it has constant coefficients or not. If the equation is linear, find the largest interval containing 1 on which the differential equation is normal.

b. Solve the initial-value problem.

2. (8 points) For the initial-value problem

$$(t-1)x' + x = 0, \quad x(1) = 0$$

a. Decide whether the Existence-and-Uniqueness Theorem applies.

b. Decide whether there is a solution or not, and if there is, decide whether it is unique. Give reasons.

3. (15 points)

a. Find the general solution of $(D^2+4)(D^2-4)(D-4)^2x = 0$.

b. Find the general solution of $(D-3)^2x = e^{3t}$.

c. Is $x(t) = c_1 + c_2e^t$ the general solution of $(D^2 - D)x = 0$?

4. (5 points) For each of the following two vector functions decide whether it is a solution of the second-order homogeneous system $D\vec{x} = \begin{pmatrix} 0 & 1 \\ -6/t^2 & 4/t \end{pmatrix}\vec{x}$.

$$\vec{h}_1(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix} \qquad \vec{h}_2(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix}$$

5. (12 points) Find the general solution of $D\vec{x} = A\vec{x}$, where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6. (8 points) Find the general solution of $D\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

You may use that the general solution of the associated homogeneous system is $c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.

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7. (8 points)

a. Find the Laplace transform of $f(t) = \begin{cases} 0 & t < \pi \\ t\sin(2t) & t \geq \pi \end{cases}$.

b. Find the inverse Laplace transform of $F(s) = \dfrac{9}{s^3 + 6s^2 + 9s}$

8. (10 points) Use the Laplace transform to solve $(D^2+2D+2)^2x = 0$ with $x(0) = x'(0) = x''(0) = 0$ and $x'''(0) = 2$. *No credit will be given for a solution using any other method.*

9. (10 points)

a. Find the equivalent system to the equation $(D^2 - 2D - 3)x = 0$ and write it in matrix form, $D\vec{x} = A\vec{x}$.

b. Draw the phase portrait for the system $D\vec{x} = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}\vec{x}$ and state whether the origin is a stable or unstable equilibrium point.

10. (14 points) Consider the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = e^x y - x$$

a. Find all equilibria.

b. Show that $E(x, y) = -x^2 - y^2$ is a Lyapunov function for this system.

c. Classify each equilibrium as an attractor, a repeller, or neither of these.

d. Determine the stability of each equilibrium.

e. Draw the phase portrait of the *linearization* of each equilibrium.

f. For each equilibrium determine whether the Hartman–Grobman Theorem applies.

g. Show that this system of differential equations has no closed integral curve.

END OF EXAMINATION