

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!

Remember to sign your blue book.

With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (10 points) For the initial-value problem

$$t \frac{dx}{dt} - x = t^3, \quad x(1) = 0$$

- a. State the order of the differential equation, whether the differential equation is linear, homogeneous or nonhomogeneous, and whether it has constant coefficients or not. If the equation is linear, find the largest interval containing 1 on which the differential equation is normal.

Solution: First order, linear, non-homogeneous, not constant coefficients. Normal on $t > 0$.

- b. Solve the initial-value problem.

Solution: First solve homogeneous equation: $t \frac{dx}{dt} = x$ giving $x(t) = kt$ (either by inspection or separation of variables). Then let $x(t) = k(t)t$. Plug into equation giving $k'(t)t^2 = t^3$, so $x(t) = \frac{t^3}{2} + ct$. The initial condition gives $x(1) = \frac{1}{2}(t^3 - t)$.

2. (8 points) For the initial-value problem

$$(t - 1)x' + x = 0, \quad x(1) = 0$$

- a. Decide whether the Existence-and-Uniqueness Theorem applies.

Solution: It does not: The right-hand side of $x' = -x/(t - 1)$ is discontinuous at $t = 1$.

- b. Decide whether there is a solution or not, and if there is, decide whether it is unique. Give reasons.

Solution: $x(t) = 0$ is a solution by inspection, so there is a solution. It is unique: Separation of variables shows that any solution that is not zero for all values of t is of the form $k/(t - 1)$, and only $k = 0$ gives a solution defined for $t = 1$.

3. (15 points)

- a. Find the general solution of $(D^2 + 4)(D^2 - 4)(D - 4)^2x = 0$.

Solution: Using the formula, $x(t) = c_1 \sin(2t) + c_2 \cos(2t) + c_3 e^{2t} + c_4 e^{-2t} + c_5 e^{4t} + c_6 t e^{4t}$.

- b. Find the general solution of $(D - 3)^2x = e^{3t}$.

Solution: General solution of $(D - 3)^2x = 0$ is $x(t) = c_1 e^{3t} + c_2 t e^{3t}$. Annihilator of e^{3t} is $(D - 3)$, so make simplified guess that $p(t) = kt^2 e^{3t}$. Substitution gives $e^{3t} \stackrel{!}{=} (D - 3)^2 kt^2 e^{3t} = e^{3t} D^2 kt^2 = 2k e^{3t}$, so $x(t) = c_1 e^{3t} + c_2 t e^{3t} + \frac{t^2}{2} e^{3t}$.

- c. Is $x(t) = c_1 + c_2 e^t$ the general solution of $(D^2 - D)x = 0$?

Solution: Yes it is. $r^2 - r$ has roots 0 and 1, and the Wronskian is nonzero: $\det \begin{pmatrix} 1 & e^t \\ 0 & e^t \end{pmatrix} = e^t \neq 0$.

4. (5 points) For each of the following two vector functions decide whether it is a solution of the second-order homogeneous system $D\vec{x} = \begin{pmatrix} 0 & 1 \\ -6/t^2 & 4/t \end{pmatrix} \vec{x}$.

$$\vec{h}_1(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix} \quad \vec{h}_2(t) = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix}$$

Solution: $Dh_1 = \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6/t^2 & 4/t \end{pmatrix} h_1(t)$, $Dh_2 = \begin{pmatrix} 3t^2 \\ 6t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6/t^2 & 4/t \end{pmatrix} h_2(t)$,

5. (12 points) Find the general solution of $D\vec{x} = A\vec{x}$, where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution: By inspection or computation, the eigenvalues are $\pm i, 0, 0$.

$$A - iI = \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix} \rightarrow \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for the eigenvalue i , which gives the solution $e^{it} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix}$.

$$(A - 0 \cdot I)^2 = A^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which gives generalized eigenvectors $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. The general solution is

$$\begin{aligned} c_1 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + c_3 e^{0 \cdot t} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + tA \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_4 e^{0 \cdot t} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + tA \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ = c_1 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ t \\ 1 \end{pmatrix}. \end{aligned}$$

6. (8 points) Find the general solution of $D\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

You may use that the general solution of the associated homogeneous system is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.

Solution: One can note by inspection that $\vec{p}(t) = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a particular solution. Otherwise, note (inspection, row-reduction or Cramer's rule) that $c_1'(t) = 0$ and $c_2'(t) = e^{-2t}$ are solutions of $c_1'(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2'(t) \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives $c_1(t) = 0$ and $c_2(t) = -\frac{1}{2}e^{-2t}$, and $\vec{p}(t) = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the general solution is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

7. (8 points)

a. Find the Laplace transform of $f(t) = \begin{cases} 0 & t < \pi \\ t \sin(2t) & t \geq \pi \end{cases}$.

Solution: $f(t) = u_\pi(t)t \sin(2t)$ so, $\mathcal{L}[f(t)] = e^{-\pi s} \mathcal{L}[(t + \pi) \sin(2(t + \pi))]$. Since $\sin(2(t + \pi)) = \sin(2t + 2\pi)$, $\mathcal{L}[f(t)] = e^{-\pi s} \mathcal{L}[(t + \pi) \sin(2t)]$. From the formula, $\mathcal{L}[\pi \sin(2t)] = \frac{2\pi}{s^2 + 4}$. Using the second differentiation formula, $\mathcal{L}[t \sin(2t)] = -\frac{d}{ds} \left(\frac{2}{s^2 + 4} \right)$. Thus, $\mathcal{L}[f(t)] = e^{-\pi s} \left(\frac{4s}{(s^2 + 4)^2} + \frac{2\pi}{s^2 + 4} \right)$.

b. Find the inverse Laplace transform of $F(s) = \frac{9}{s^3 + 6s^2 + 9s}$

Solution: Writing $s^3 + 6s^2 + 9s = s(s + 3)^2$, we get $\frac{9}{s^3 + 6s^2 + 9s} = \frac{A}{s} + \frac{B}{s + 3} + \frac{C}{(s + 3)^2} = \frac{(A + B)s^2 + (6A + 3B + C)s + 9A}{s^3 + 6s^2 + 9s}$. This yields $A = 1, B = -1, C = -3$, so $\frac{9}{s^3 + 6s^2 + 9s} = \frac{1}{s} - \frac{1}{s + 3} - \frac{3}{(s + 3)^2}$. To find $\mathcal{L}^{-1} \left[\frac{3}{(s + 3)^2} \right] = e^{-3t} \mathcal{L}^{-1} \left[\frac{3}{s^2} \right] = 3te^{-3t}$, so $\mathcal{L}^{-1}[F(s)] = 1 - e^{-3t} - 3te^{-3t}$.

8. (10 points) Use the Laplace transform to solve $(D^2 + 2D + 2)^2 x = 0$ with $x(0) = x'(0) = x''(0) = 0$ and $x'''(0) = 2$. No credit will be given for a solution using any other method.

Solution: $\mathcal{L}[(D^2 + 2D + 2)^2 x] = (s^2 + 2s + 2)^2 \mathcal{L}[x] - 2 = 0$, so $\mathcal{L}[x] = \frac{2}{((s+1)^2+1)^2}$. Thus, $x(t) = e^{-t} \mathcal{L}^{-1} \left[\frac{2}{(s^2+1)^2} \right] = 2e^{-t}(\sin(t) * \sin(t)) = e^{-t}(\sin(t) - t \cos(t))$.

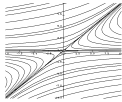
9. (10 points)

a. Find the equivalent system to the equation $(D^2 - 2D - 3)x = 0$ and write it in matrix form, $D\vec{x} = A\vec{x}$.

Solution: $D\vec{x} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \vec{x}$.

b. Draw the phase portrait for the system $D\vec{x} = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} \vec{x}$ and state whether the origin is a stable or unstable equilibrium point.

Solution: A has eigenvalues 2 and -1 . This gives a saddle, which is unstable. Eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\lambda = 2$ and

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = -1$. 

10. (14 points) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= e^x y - x \end{aligned}$$

a. Find all equilibria.

Solution: $(0, 0)$ only.

b. Show that $E(x, y) = -x^2 - y^2$ is a Lyapunov function for this system.

Solution: $(\partial E / \partial x)x' + (\partial E / \partial y)y' = -2xy - 2e^x y^2 + 2xy = -2e^x y^2 < 0$ for $y \neq 0$.

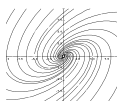
c. Classify each equilibrium as an attractor, a repeller, or neither of these.

Solution: Note that at the origin $(\partial^2 E / \partial x^2)(\partial^2 E / \partial y^2) - (\partial^2 E / \partial x \partial y)^2 = 4$ and $\partial^2 E / \partial x^2 = -2$, so $(0, 0)$ is a global maximum for the Lyapunov function, hence a repeller. **Or:** The linearization at the origin is $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, and it has eigenvalues $(1 \pm \sqrt{3}i)/2$, so $(0, 0)$ is a repeller by the Hartman–Grobman Theorem.

d. Determine the stability of each equilibrium.

Solution: Being a repeller, the origin is unstable.

e. Draw the phase portrait of the *linearization* of each equilibrium.

Solution: Outward spirals, clockwise because $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. 

f. For each equilibrium determine whether the Hartman–Grobman Theorem applies.

Solution: It does, since both eigenvalues have real part $1/2 \neq 0$.

g. Show that this system of differential equations has no closed integral curve.

Solution: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = e^x > 0$ everywhere. **Or:** $2rr' = 2xy + 2y(e^x y - x) = 2e^x y^2 > 0$ for $y \neq 0$, so $r = \sqrt{x^2 + y^2}$ is strictly increasing. **Or:** The presence of a Lyapunov function makes this impossible.