Mathematics 38 **Differential Equations**
Final Examination **May 5, 8:30–10:30**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.

Show all work and cross out work you do not want graded!

Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (15 points)

a. Find the general solution of $D\vec{x} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

You may use that the general solution of the associated homogeneous system is $c_1 \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.

b. Find the general solution of $x'' + x = \sin 3t$.

c. Find the general solution of $tx' = (x+1)(t^2+1)$ for $t > 0$.

2. (5 points) Calculate the inverse Laplace transform of $\dfrac{s}{3(s^2+9)^2}$.

3. (10 points) Let $f(t) = \begin{cases} 0 & t < \pi/6 \\ \sin 3t & \pi/6 \le t < \pi/3 \\ 0 & t \ge \pi/3 \end{cases}$.

a. Express $f(t)$ in step-function notation.

b. Find the Laplace transform of $f(t)$.

4. (10 points) Use the Laplace transform to solve $(D-2)(D-1)^2 x = -2e^t$ with $x(0) = x'(0) = 0$ and $x''(0) = 2$.

5. (5 points) $c_1(t^3+t^2) + c_2(t^2+1) + c_3(t^3-1) + c_4 t + t^4$ is a solution of $D^4 x = 24$. Decide whether this is the general solution.

6. (15 points) Given the system

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= x^2 + 2xy - 8x \\ \frac{dy}{dt} &= 3xy + y^2 - 9y \end{aligned}$$

a. Find the equilibrium points.

b. Find the linearization matrix $A_{(x,y)}$ of (S).
[Check your work carefully! You will not get credit for the following parts if $A_{(x,y)}$ is wrong.]

c. Classify each equilibrium as an attractor, a repeller or neither of these.

d. Classify each equilibrium as stable or unstable.

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7. (15 points) Consider the function $E(x,y) = x^2y - xy^2 + 3xy$ and the system

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= x^2 + 3x - 2xy \\ \frac{dy}{dt} &= y^2 - 3y - 2xy. \end{aligned}$$

a. Verify that E is a constant of motion for (S).

b. Find the equilibria of (S).

c. Find the critical points of E, and classify them as extremum (that is, maximum or minimum) or saddle.

d. Classify the equilibria of (S) as stable or unstable.

8. (5 points) Draw the phase portrait of $x' = x^2 + 2x + 2$.

9. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.

10. (10 points) Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

You may use the following information without verifying it: The eigenvalues of A are 0 and -1 and

$$\begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$$

are linearly independent solutions of $D\vec{x} = A\vec{x}$. (This information makes the problem easy!!)

END OF EXAMINATION