Mathematics 38 — **Differential Equations**
Final Examination — **May 5, 8:30–10:30**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight.
Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

1. (15 points)

a. Find the general solution of $D\vec{x} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

You may use that the general solution of the associated homogeneous system is $c_1 \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.

Solution: One can note by inspection that $\vec{p}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ is a particular solution. Otherwise, note (by inspection, row-reduction or Cramer's rule) that $c_1'(t) = 0$ and $c_2'(t) = e^{-t}$ are solutions of $c_1'(t) \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} + c_2'(t) \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$. This gives $c_2(t) = -e^{-t}$ and $\vec{p}(t) = -\begin{pmatrix} e^t \\ e^t \end{pmatrix}$, so the general solution is $c_1 \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

b. Find the general solution of $x'' + x = \sin 3t$.

Solution: The general solution of the associated homogeneous differential equation in $c_1 \cos t + c_2 \sin t$. The method of undetermined coefficients gives the simplified guess $a \cos 3t + b \sin 3t$. Plug in to get $\sin 3t = (-9a \cos 3t - 9b \sin 3t) + (a \cos 3t + b \sin 3t)$, so $a = 0$, $b = -1/8$. The general solution is $c_1 \cos t + c_2 \sin t - (1/8) \sin 3t$.

c. Find the general solution of $tx' = (x + 1)(t^2 + 1)$ for $t > 0$.

Solution: Separation of variables gives $\ln(x+1) = \int \frac{dx}{x+1} = \int t + t^{-1}\, dt = t^2/2 + \ln t + C$, so $x(t) = kte^{t^2/2} - 1$.

2. (5 points) Calculate the inverse Laplace transform of $\frac{s}{3(s^2+9)^2}$.

Solution: $\mathscr{L}^{-1}[\frac{1}{9}\frac{3}{s^2+9}\frac{s}{s^2+9}] = \frac{1}{9} \sin 3t * \cos 3t = \frac{t}{18} \sin 3t$.

3. (10 points) Let $f(t) = \begin{cases} 0 & t < \pi/6 \\ \sin 3t & \pi/6 \le t < \pi/3 \\ 0 & t \ge \pi/3 \end{cases}$.

a. Express $f(t)$ in step-function notation.

Solution: $(u_{\pi/6}(t) - u_{\pi/3}) \sin 3t$.

b. Find the Laplace transform of $f(t)$.

Solution: $e^{-s\frac{\pi}{6}} \mathscr{L}[\sin(3(t+\frac{\pi}{6}))] - e^{-2\frac{\pi}{3}} \mathscr{L}[\sin(3(t+\frac{\pi}{3}))] = e^{-s\frac{\pi}{6}} \mathscr{L}[\cos 3t] + e^{-s\frac{\pi}{3}} \mathscr{L}[\sin 3t] = \frac{se^{-s\frac{\pi}{6}} + 3e^{-s\frac{\pi}{3}}}{s^2+9}$.

4. (10 points) Use the Laplace transform to solve $(D-2)(D-1)^2 x = -2e^t$ with $x(0) = x'(0) = 0$ and $x''(0) = 2$.

Solution: Laplace-transform to $(s-2)(s-1)^2 \mathscr{L}[x] - 2 = \frac{-2}{s-1}$, which becomes

$$\mathscr{L}[x] = \frac{2}{(s-2)(s-1)^2} - \frac{2}{(s-2)(s-1)^3} = \frac{2(s-1)-2}{(s-2)(s-1)^3} = \frac{2(s-2)}{(s-2)(s-1)^3} = \frac{2}{(s-1)^3},$$

so

$$x(t) = 2\mathscr{L}^{-1}[\frac{1}{(s-1)^3}] = 2e^t \mathscr{L}^{-1}[\frac{1}{s^3}] = t^2 e^t.$$

5. (5 points) $c_1(t^3+t^2)+c_2(t^2+1)+c_3(t^3-1)+c_4t+t^4$ is a solution of $D^4x=24$. Decide whether this is the general solution.

Solution: It is not. Either note that $(t^2+1)+(t^3-1)=t^3+t^2$ or compute the Wronskian at $t=0$:

$$\det\begin{pmatrix} t^3+t^2 & t^2+1 & t^3-1 & t \\ 3t^2+2t & 2t & 3t^2 & 1 \\ 6t+2 & 2 & 6t & 0 \\ 6 & 0 & 6 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 6 & 0 & 6 & 0 \end{pmatrix} = 12\det\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 12\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 0$$

or compute the Wronskian using row operations:

$$\det\begin{pmatrix} t^3+t^2 & t^2+1 & t^3-1 & t \\ 3t^2+2t & 2t & 3t^2 & 1 \\ 6t+2 & 2 & 6t & 0 \\ 6 & 0 & 6 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & -1 & t \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 0.$$

6. (15 points) Given the system

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= x^2+2xy-8x \\ \frac{dy}{dt} &= 3xy+y^2-9y \end{aligned}$$

a. Find the equilibrium points.

Solution: $0=\dfrac{dx}{dt}=x^2+2xy-8x=x(x+2y-8)$ and $0=\dfrac{dy}{dt}=3xy+y^2-9y=y(3x+y-9)$ gives the equilibria $(0,0)$, $(0,9)$, $(8,0)$, $(2,3)$.

b. Find the linearization matrix $A_{(x,y)}$ of (S).
[Check your work carefully! You will not get credit for the following parts if $A_{(x,y)}$ is wrong.]

Solution: $A_{(x,y)}=\begin{pmatrix} 2x+2y-8 & 2x \\ 3y & 3x+2y-9 \end{pmatrix}$.

c. Classify each equilibrium as an attractor, a repeller or neither of these.

Solution: $A_{(0,0)}=\begin{pmatrix} -8 & 0 \\ 0 & -9 \end{pmatrix}$, $\lambda=-8,-9$, attractor,

$A_{(0,9)}=\begin{pmatrix} 10 & 0 \\ 27 & 9 \end{pmatrix}$, $\lambda=10,9$, repeller,

$A_{(8,0)}=\begin{pmatrix} 8 & 16 \\ 0 & 15 \end{pmatrix}$, $\lambda=8,15$, repeller,

$A_{(2,3)}=\begin{pmatrix} 2 & 4 \\ 9 & 3 \end{pmatrix}$, $\lambda=\dfrac{5\pm\sqrt{25+120}}{2}\approx\dfrac{5\pm\sqrt{144}}{2}=\dfrac{5}{2}\pm 6$, saddle, neither attractor nor repeller.

d. Classify each equilibrium as stable or unstable.

Solution: $(0,0)$: attractor, stable; $(0,9)$: repeller, unstable; $(8,0)$: repeller, unstable; $(2,3)$: neither, unstable.

7. (15 points) Consider the function $E(x,y)=x^2y-xy^2+3xy$ and the system

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= x^2+3x-2xy \\ \frac{dy}{dt} &= y^2-3y-2xy. \end{aligned}$$

a. Verify that E is a constant of motion for (S).

Solution: $\dfrac{d}{dt}E(x(t),y(t))=\dfrac{\partial E}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial E}{\partial y}\dfrac{dy}{dt}=\dfrac{\partial}{\partial x}[x^2y-xy^2+3xy]\dfrac{dx}{dt}+\dfrac{\partial}{\partial y}[x^2y-xy^2+3xy]\dfrac{dy}{dt}=[2xy-y^2+3y](x^2+3x-2xy)+[x^2-2xy+3x](y^2-3y-2xy)=0$.

b. Find the equilibria of (S).

Solution: $0=x^2+3x-2xy=x(x+3-2y)$ and $0=y^2-3y-2xy=y(y-3-2x)$ gives the equilibria $(0,0)$, $(0,3)$, $(-3,0)$ and $(-1,1)$.

c. Find the critical points of E, and classify them as extremum (that is, maximum or minimum) or saddle.

Solution: $0 = \dfrac{\partial E}{\partial x} = 2xy - y^2 + 3y = y(2x - y + 3)$ and $0 = \dfrac{\partial E}{\partial y} = x^2 - 2xy + 3x = x(x - 2y + 3)$ gives critical points $(0,0)$, $(0,3)$, $(-3,0)$ and $(-1,1)$. The discriminant of the Hessian is $\Delta(x,y) = \dfrac{\partial^2 E}{\partial x^2}\dfrac{\partial^2 E}{\partial y^2} - (\dfrac{\partial^2 E}{\partial x \partial y})^2 = [2y][-2x] - [2x - 2y + 3]^2$. Thus we get

$$\begin{aligned}
(0,0) &: \Delta(0,0) = -3^2 < 0, \text{ saddle},\\
(0,3) &: \Delta(0,3) = -3^2 < 0, \text{ saddle},\\
(-3,0) &: \Delta(-3,0) = -3^2 < 0, \text{ saddle},\\
(-1,1) &: \Delta(-1,1) = 2 \cdot 2 - 1^2 = 3 > 0, \text{ extremum}.
\end{aligned}$$

d. Classify the equilibria of (S) as stable or unstable.

Solution: From the previous parts we conclude

$$\begin{aligned}
(0,0) &: \text{ unstable},\\
(0,3) &: \text{ unstable},\\
(-3,0) &: \text{ unstable},\\
(-1,1) &: \text{ stable}.
\end{aligned}$$

8. (5 points) Draw the phase portrait of $x' = x^2 + 2x + 2$.

Solution: $x' > 0$ so the phase portrait is $\rightarrowtail$.

9. (10 points) Show that any set of vectors that includes $\vec{0}$ is linearly dependent.

Solution: Denote the vectors by $\vec{0}, \vec{v}_1, \ldots, \vec{v}_n$ and note that in

$$1 \cdot \vec{0} + 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0}$$

not all coefficients are zero.

10. (10 points) Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

You may use the following information without verifying it: The eigenvalues of A are 0 and -1 and

$$\begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix}$$

are linearly independent solutions of $D\vec{x} = A\vec{x}$. (This information makes the problem easy!!)

Solution: The given solutions correspond to the eigenvalue 0, and there are 3 of them, so we only need to add at most two solutions coming from the eigenvalue -1. By inspection of A (or a minimal computation), $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are *actual* eigenvectors for the eigenvalue -1. This gives the general solution

$$c_1 \begin{pmatrix} 1 + \frac{t^2}{2} \\ t \\ 1 \\ t^2/2 \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t \\ 1 \\ 0 \\ t \\ 1 \end{pmatrix} + c_4 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

END OF EXAMINATION