Tufts UNIVERSITY

Mathematics 38 **Differential Equations**

Final Examination **December 16, 8:30–10:30**

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!

Remember to sign your blue book.

With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

Please write the answers to problems 1–8 on the cover of the blue book in the corresponding box, as shown below.

1	Yes
2	w.
3	e.
4	c.
5	a.n.
6	cos(t)
7	high
8	Yes
9	
10	
T	

1. (3 points, answer only, no partial credit) Can the system below be solved by Cramer's rule?

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ -x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

Solution: **No**, the determinant of the matrix of coefficients is zero.

2. (3 points, answer only, no partial credit) Consider the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ -x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

Choose one answer: The system has

a. a unique solution, **b.** no solution, **c.** more than 1 solution. **d.** None of the above.

Solution: **c.**. (**d.** is impossible, **a.** is impossible because the previous problem shows that Cramer's test gives a zero determinant, and **b.** is not true because $x_1 = x_2 = x_3 = 0$ is an obvious solution.)

3. (3 points, answer only, no partial credit) Choose one answer.

$$\det \begin{pmatrix} 5 & 1 & \sin t & t^2+3 & 1 \\ 0 & 4 & e^t & e^t & 0 \\ 0 & 0 & 3 & \ln t & 8 \\ 0 & 0 & 0 & 2 & \sqrt{t} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

a. 5, **b.** $2\sqrt{t}$, **c.** 120, **d.** $120 - \sin t - \frac{1}{4}e^t - \frac{1}{4}(t^2+3)e^t$. **e.** None of the above.

Solution: **c.**. This is easy because the matrix is triangular.

4. (3 points, answer only, no partial credit) Choose one answer.

The functions e^t, e^{-t}, e^{t+1} are

a. linearly dependent **b.** linearly independent. **c.** None of the above.

Solution: **a.** (because $e \cdot e^t + 0 \cdot e^{-t} - 1 \cdot e^{t+1} = 0$ for all t.)

5. (3 points, answer only, no partial credit) Consider the differential equation

$$t^4 x'' - 2xx' = 0.$$

Choose *every* function from the following that is a solution of this differential equation.

a. 0, **b.** 1, **c.** t, **d.** t^2, **e.** t^3, **f.** t^4, **g.** None of the above.

Solution: **a., b., e.**: Plug them in . . .

6. (3 points, answer only, no partial credit) Suppose $g(t)$ is a continuous function. Solve the initial-value problem $x' = g(t)x, \quad x(1) = 0$.

(Hint: Think before applying standard techniques.)

Solution: $x(t) = 0$ (for all t). By inspection this is a solution of the ODE (since $\frac{dx}{dt} = 0$), and it satisfies the initial condition. Since the theorem about existence and uniqueness applies, any different answer is wrong.

One could get this by separation of variables (not recommended): We separate variables to give $\frac{dx}{x} = g(t)dt$ (if $x \neq 0$). Integrating (and forgetting about absolute values), we get $\ln x = \int g(t)\,dt + C$, so $x = e^{C + \int g(t)\,dt} = Ae^{\int g(t)\,dt}$ for some constant A (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we can take $A = 0$, which gives $x(t) = 0$ for all t.

7. (3 points, answer only, no partial credit) For the differential equation

$$t\frac{dx}{dt} - x = t^3$$

state the order of the differential equation, whether the differential equation is linear, homogeneous or nonhomogeneous, and whether it has constant coefficients or not. If the equation is linear, find the largest interval containing 1 on which the differential equation is normal.

Solution: First order, linear, nonhomogeneous, not constant coefficients. Normal for $t > 0$.

8. (3 points, answer only, no partial credit) For the initial-value problem

$$(t-1)x' + x = 0, \quad x(1) = 0$$

decide whether the Existence-and-Uniqueness Theorem applies.

Solution: It does not: The right-hand side of $x' = -x/(t-1)$ is discontinuous at $t = 1$.

9. (6 points)

a. Find the general solution of $(D^2+4)(D^2-4)(D-4)^2x = 0$.

Solution: Using the formula, $x(t) = c_1\sin(2t) + c_2\cos(2t) + c_3e^{2t} + c_4e^{-2t} + c_5e^{4t} + c_6te^{4t}$.

b. Find a simplified guess for a particular solution of $(D-3)^2x = e^{3t}$.

Solution: General solution of $(D-3)^2x = 0$ is $x(t) = c_1e^{3t} + c_2te^{3t}$. Annihilator of e^{3t} is $(D-3)$, so make simplified guess that $p(t) = kt^2e^{3t}$.

10. (7 points) Find the general solution of $D\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

You may use that the general solution of the associated homogeneous system is $c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.

Solution: One can note by inspection that $\vec{p}(t) = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a particular solution. Otherwise, note (inspection, row-reduction or Cramer's rule) that $c_1'(t) = 0$ and $c_2'(t) = e^{-2t}$ are solutions of $c_1'(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2'(t)\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives $c_1(t) = 0$ and $c_2(t) = -\frac{1}{2}e^{-2t}$, and $\vec{p}(t) = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the general solution is $c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

11. (8 points)

a. Find the Laplace transform of $f(t) = \begin{cases} 0 & t < \pi \\ t\sin(2t) & t \geq \pi \end{cases}$.

Solution: $f(t) = u_\pi(t)t\sin(2t)$ so, $\mathscr{L}[f(t)] = e^{-\pi s}\mathscr{L}[(t+\pi)\sin(2(t+\pi))]$. Since $\sin(2(t+\pi)) = \sin(2t+2\pi)$, $\mathscr{L}[f(t)] = e^{-\pi s}\mathscr{L}[(t+\pi)\sin(2t)]$. From the formula, $\mathscr{L}[\pi\sin(2t)] = \frac{2\pi}{s^2+4}$. Using the second differentiation formula, $\mathscr{L}[t\sin(2t)] = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$. Thus, $\mathscr{L}[f(t)] = e^{-\pi s}\left(\frac{4s}{(s^2+4)^2} + \frac{2\pi}{s^2+4}\right)$.

b. Find the inverse Laplace transform of $F(s) = \dfrac{9}{s^3+6s^2+9s}$

Solution: Writing $s^3+6s^2+9s = s(s+3)^2$, we get $\frac{9}{s^3+6s^2+9s} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{(s+3)^2} = \frac{(A+B)s^2+(6A+3B+C)s+9A}{s^3+6s^2+9s}$. This yields $A=1$, $B=-1$, $C=-3$, so $\frac{9}{s^3+6s^2+9s} = \frac{1}{s} - \frac{1}{s+3} - \frac{3}{(s+3)^2}$. To find $\mathscr{L}^{-1}\left[\frac{3}{(s+3)^2}\right] = e^{-3t}\mathscr{L}^{-1}\left[\frac{3}{s^2}\right] = 3te^{-3t}$, so $\mathscr{L}^{-1}[F(s)] = 1 - e^{-3t} - 3te^{3t}$.

12. (10 points) Use the Laplace transform to solve $(D^2+2D+2)^2x = 0$ with $x(0) = x'(0) = x''(0) = 0$ and $x'''(0) = 2$. *No credit will be given for a solution using any other method.*

Solution: $\mathscr{L}[(D^2+2D+2)^2x] = (s^2+2s+2)^2\mathscr{L}[x] - 2 = 0$, so $\mathscr{L}[x] = \frac{2}{((s+1)^2+1)^2}$. Thus, $x(t) = e^{-t}\mathscr{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = 2e^{-t}(\sin(t) * \sin(t)) = e^{-t}(\sin(t) - t\cos(t))$.

13. (10 points) Find the general solution of the system

$$
\begin{aligned}
x_1' &= x_1 - x_2 \\
x_2' &= -x_2 + 3x_4 \\
x_3' &= x_3 \\
x_4' &= -x_1 + x_2
\end{aligned}
$$

Without verifying it, you may use that

$$
\begin{aligned}
x_1 &= 3c_1 - c_2e^{2t} - c_3e^{-2t} \\
x_2 &= 3c_1 + c_2e^{2t} - 3c_3e^{-2t} \\
x_3 &= 0 \\
x_4 &= c_1 + c_2e^{2t} + 3c_3e^{-2t}
\end{aligned}
$$

is a solution for any choice of c_1, c_2, c_3.

Solution: By inspection, $x_3 = c_4e^t$ is a solution (of $x_3' = x_3$, which is completely decoupled from the other equations), so one finds

$$
\begin{aligned}
x_1 &= 3c_1 - c_2e^{2t} - c_3e^{-2t} \\
x_2 &= 3c_1 + c_2e^{2t} - 3c_3e^{-2t} \\
x_3 &= c_4e^t \\
x_2 &= c_1 + c_2e^{2t} + 3c_3e^{-2t}
\end{aligned}
$$

This can also be obtained by converting to a system and finding an obvious eigenvalue/eigenvector pair by inspection: The corresponding system $D\vec{x} = A\vec{x}$ has

$$
A = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 1 & * \\ * & * & 0 & * \end{pmatrix}.
$$

14. (20 points) Given the system

$$(S) \qquad \begin{aligned} \frac{dx}{dt} &= x - x^2 - xy \\ \frac{dy}{dt} &= 2xy - y \end{aligned}$$

a. Find the equilibrium points.

Solution: $0 = \dfrac{dx}{dt} = x - x^2 - xy = x(1-x-y)$ and $0 = \dfrac{dy}{dt} = 2xy - y = y(2x-1)$ gives the equilibria $(0,0)$, $(1,0)$, $(1/2, 1/2)$.

b. Find the linearization matrix $A_{(x,y)}$ of (S).
[Check your work carefully! You will not get credit for the following parts if $A_{(x,y)}$ is wrong.]

Solution: $A_{(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ 2y & 2x-1 \end{pmatrix}$.

c. Classify each equilibrium as an attractor, a repeller or neither of these.

Solution: $A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\lambda = -1, 1$, Hartman–Grobman applies, hence **saddle, neither** attractor nor repeller,
$A_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$, $\lambda = -1, 1$, Hartman–Grobman applies, hence **saddle, neither** attractor nor repeller,
$A_{(1/2,1/2)} = \begin{pmatrix} -1/2 & -1/2 \\ 1 & 0 \end{pmatrix}$, $\lambda = \dfrac{-1 \pm i\sqrt{7}}{4}$, Hartman–Grobman applies, hence **attractor**.

d. Classify each equilibrium as stable or unstable.

Solution: $(0,0)$**: neither, unstable;** $(1,0)$**: neither, unstable;** $(1/2,1/2)$**: attractor, stable.**

e. Draw the phase portrait.

Solution: The only nontrivial additional information is the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ of $A_{(1,0)}$. See p. 360 of the textbook.

15. (15 points) Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x^3 + y \\ \frac{dy}{dt} &= y^3 - x \end{aligned}$$

a. Find all equilibria.

Solution: $(0,0)$ only.

b. Decide whether $E(x,y) = -x^2 - y^2$ is a constant of motion for this system.

Solution: No: $dE/dt = (\partial E/\partial x)x' + (\partial E/\partial y)y' = -2x(x^3+y) - 2y(y^3-x) = -2(x^4+y^4) < 0$ for $(x,y) \neq \vec{0}$.

c. Decide whether $E(x,y) = -x^2 - y^2$ is a Lyapunov function for this system.

Solution: Yes, since from above, $dE/dt < 0$ for $(x,y) \neq \vec{0}$.

d. Classify each equilibrium as an attractor, a repeller, or neither of these.

Solution: By inspection, $(0,0)$ is a global maximum for the Lyapunov function, hence a **repeller**.

e. Determine the stability of each equilibrium.

Solution: Being a repeller, the origin is **unstable**.

f. Draw the phase portrait of the *linearization* of each equilibrium.

Solution: Circles, clockwise because $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

g. For each equilibrium determine whether the Hartman–Grobman Theorem applies.

Solution: No, it does not, since both eigenvalues are purely imaginary.

h. Decide whether this system of differential equations has a closed integral curve.

Solution: No, the presence of a Lyapunov function makes this impossible. **Or:** $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = x^2 + y^2 > 0$ everywhere except $\vec{0}$. **Or:** $2rr' = 2x(x^3+y) + 2y(y^3-x) = 2(x^4+y^4) > 0$ for $(x,y) \neq \vec{0}$, so $r = \sqrt{x^2+y^2}$ is strictly increasing.

END OF EXAMINATION