Tufts University

Math 38 | Department of Mathematics | May 4, 2009

Final Exam

No calculators, notes, books, pagers, mobile phones or other electronic devices are allowed on the exam. All answers should be in terms of real numbers and functions. You must **show all your work** to receive credit. *You are required to sign your exam book. With your signature, you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (12 points) Consider the equation

$$\text{(E)} \qquad \frac{dx}{dt} - tx = t$$

   (a) Decide whether (i) the equation is in standard form; find the largest interval containing 1 on which the equation is normal.

   (ii) the equation is linear

   (ii) the equation is homogeneous

   (b) Does (E) satisfy the hypotheses of the existence and uniqueness theorem at

   (i) the point $t = 0$, $x = 1$?

   (ii) the point $t = 1$, $x = 0$?

   (c) Solve the initial value problem $\frac{dx}{dt} - tx = t$, $\quad x(1) = 0$ or prove that it has no solution.

2. (12 points)

   (a) Find the general solution of $(D^2 + 1)^2(D^2 - 1)D^2x = 0$

   (b) Find the general solution of $D^2(D + 1)x = e^{-t}$

Exam continues on other side

(c) Decide whether $c_1 \begin{pmatrix} -e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} te^{-t} \\ -e^{-t} \\ -te^{-t} \end{pmatrix} + c_3 \begin{pmatrix} (3t-2)e^{-t} \\ -3e^{-t} \\ -(3t-2)e^{-t} \end{pmatrix}$

is a solution of $D\mathbf{x} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x}$

(d) Decide whether the expression in (c) is the general solution of the equation.

3. (14 points) Find the general solution of $D\mathbf{x} = \begin{pmatrix} 0 & 8 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -8 & 0 \end{pmatrix} \mathbf{x}$

4. (14 points) The general solution of

$D\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$ is $c_1 e^{2t} \begin{pmatrix} 1-t \\ t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -t \\ 1+t \end{pmatrix}$

You do not have to verify this.

Use this fact to solve $D\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$

5. (12 points) Consider

$$f(t) = \begin{cases} t^2 & t < 1 \\ t^2 - 1 & 1 \le t < 2 \\ t^2 - 2 & 2 \le t \end{cases}$$

(a) Write f as an expression in step-function notation.

(b) Compute the Laplace transform of $f(t)$.

6. (12 points)

(a) Find the inverse Laplace transform of $\dfrac{1}{s^2 + 4s + 3}$

(b) Find the Laplace transform of $t^2 e^{3t}$

(c) Solve the initial value problem $(D+1)x = e^{-t}, \quad x(0) = 1$ using Laplace transforms.

No credit by any other method.

Exam continues on next page

7. (5 points) Show that if two vectors in a set are equal then the set of vectors are linearly dependent.

8. (7 points)

   (a) Write $(D^2 + 1)x = 0$ as a 2×2 system.

   (b) Solve the system.

   (c) Draw the phase portrait of the 2×2 system.

9. (12 points) Consider the system

$$\frac{dx}{dt} = x^3 + y$$

$$\frac{dy}{dt} = -x + y^3$$

   (a) Find all equilibria of the system.

   (b) Decide whether $E(x, y) = -x^2 - y^2$ is a constant of motion, a Lyapunov function (or neither) for the system.

   (c) Classify the equilibria as attractors, repellers or neither.

   (d) Show that the system has no closed integral curves.

End of Exam