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Math 38 Final Exam Solutions

#1 (E) $\frac{dx}{dt} - tx = t$

(a) (i) Equation (E) is in standard form.
+1
+1 It is normal on $-\infty < t < \infty$

+1 (ii) It is linear.

+1 (iii) It is non homogeneous.

(b) $\frac{dx}{dt} = f(t,x) = tx + t$

f and f_x are continuous and existence and uniqueness are guaranteed through every point (t_0, x) in the entire plane
$-\infty < t < \infty$, $-\infty < x < \infty$

+1 (i) Yes at (0, 1).
+1 (ii) Yes at (1, 0).

(c) $\frac{dx}{x+1} = t\,dt$

$\ln|x+1| = \frac{1}{2}t^2 + c$

+3 $x(t) = ke^{\frac{1}{2}t^2} - 1$

$x(1) = 0$
$\Rightarrow k = e^{-\frac{1}{2}}$

+3 $\boxed{x(t) = e^{\frac{1}{2}(t^2-1)} - 1}$

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#2. (a) $(D^2+1)^2(D^2-1)D^2x=0$

$$P(r)=(r^2+1)^2(r^2-1)r^2$$
$$=(r^2+1)^2(r+1)(r-1)\,r^2$$

has roots $0, 0, -1, 1, \pm i, \pm i$

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$$\boxed{\begin{aligned} x(t) = {} & c_1 + c_2 t + c_3 e^{-t} + c_4 e^{t} \\ & + c_5 \cos t + c_6 \sin t \\ & + c_7 t\cos t + c_8 t \sin t \end{aligned}}$$

(b) $D^2(D+1)x=e^{-t}$

(H) $H(t)=c_1+c_2t+c_3e^{-t}$

(H*) $H^*(t)=c_1+c_2t+c_3e^{-t}+c_4te^{-t}$

(N) $p(t)=kte^{-t}$ $\quad p''(t)=k(-2+t)e^{-t}$

$p'(t)=k(1-t)e^{-t}$ $\quad p'''(t)=k(3-t)e^{-t}$

$$D^2(D+1)p(t) = k(3-t)e^{-t}+k(-2+t)e^{-t}$$
$$= e^{-t} \Rightarrow 3k-2k=1 \Rightarrow k=1$$

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$$\boxed{x(t)=c_1+c_2t+c_3e^{-t}+te^{-t}}$$

#2 (c) $D\vec{x} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \vec{x} = A\vec{x}$

$$D\begin{bmatrix} -e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \quad A\begin{bmatrix} -e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} \quad \checkmark$$

$$D\begin{bmatrix} te^{-t} \\ -e^{-t} \\ -te^{-t} \end{bmatrix} = \begin{bmatrix} (1-t)e^{-t} \\ e^{-t} \\ (t-1)e^{-t} \end{bmatrix}, \quad A\begin{bmatrix} te^{-t} \\ -e^{-t} \\ -te^{-t} \end{bmatrix} = \begin{bmatrix} -te^{-t}+e^{-t} \\ e^{-t} \\ -e^{-t}+te^{-t} \end{bmatrix} \quad \checkmark$$

$$D\begin{bmatrix} (3t-2)e^{-t} \\ -3e^{-t} \\ -(3t-2)e^{-t} \end{bmatrix} = \begin{bmatrix} (5-3t)e^{-t} \\ 3e^{-t} \\ (-5+3t)e^{-t} \end{bmatrix}, \quad A\begin{bmatrix} (3t-2)e^{-t} \\ -3e^{-t} \\ -(3t-2)e^{-t} \end{bmatrix} = \begin{bmatrix} (-3t+2+3)e^{-t} \\ 3e^{-t} \\ (-3+3t-2)e^{-t} \end{bmatrix} \quad \checkmark$$

+3 The linear combination of the three vector functions is a solution.

(d) $$W\begin{bmatrix} -e^{-t} & te^{-t} & (3t-2)e^{-t} \\ 0 & -e^{-t} & -3e^{-t} \\ -e^{-t} & -te^{-t} & -(3t-2)e^{-t} \end{bmatrix}(0) = \begin{vmatrix} -1 & 0 & -2 \\ 0 & -1 & -3 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ -1 & 2 \end{vmatrix}$$

$$= -2-2 = -4 \neq 0$$

$\Rightarrow$ The linear combination of the three vector functions is the general solution.

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#3 $A = \begin{bmatrix} 0 & 8 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -8 & 0 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} -\lambda & 8 & 0 & 0 \\ -2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -8 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 8 \\ -2 & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 2 \\ -8 & -\lambda \end{vmatrix} = (\lambda^2+16)(\lambda^2+16)$$

$\lambda = \pm 4i$ with multiplicity two.

$\underline{\lambda = -4i}$

$$\left[\begin{array}{cccc|c} 4i & 8 & 0 & 0 & 0 \\ -2 & 4i & 0 & 0 & 0 \\ 0 & 0 & 4i & 2 & 0 \\ 0 & 0 & -8 & 4i & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & -4i & 0 & 0 & 0 \\ -2 & 4i & 0 & 0 & 0 \\ 0 & 0 & 8 & -4i & 0 \\ 0 & 0 & -8 & 4i & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -2i & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 2i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \\ 2 \end{bmatrix}$$

Two free variables.

$$\begin{bmatrix} 2i \\ 1 \\ 0 \\ 0 \end{bmatrix} (\cos 4t - i \sin 4t) = \begin{bmatrix} 2\sin 4t \\ \cos 4t \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 2\cos 4t \\ -\sin 4t \\ 0 \\ 0 \end{bmatrix}$$

#3 continued

$$\begin{bmatrix} 0 \\ 0 \\ i \\ 2 \end{bmatrix} (\cos 4t - i \sin 4t) = \begin{bmatrix} 0 \\ 0 \\ \sin 4t \\ 2\cos 4t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \cos 4t \\ -2\sin 4t \end{bmatrix}$$

$$\boxed{\vec{h}_1(t) = \begin{bmatrix} 2\sin 4t \\ \cos 4t \\ 0 \\ 0 \end{bmatrix} \quad \vec{h}_2(t) = \begin{bmatrix} 2\cos 4t \\ -\sin 4t \\ 0 \\ 0 \end{bmatrix} \quad \vec{h}_3(t) = \begin{bmatrix} 0 \\ 0 \\ \sin 4t \\ 2\cos 4t \end{bmatrix} \quad \vec{h}_4(t) = \begin{bmatrix} 0 \\ 0 \\ \cos 4t \\ -2\sin 4t \end{bmatrix}}$$

$$\boxed{\vec{x}(t) = c_1 \vec{h}_1(t) + c_2 \vec{h}_2(t) + c_3 \vec{h}_3(t) + c_4 \vec{h}_4(t)}$$

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#4 By Variation of Parameters,

$$\left[\begin{array}{cc|c}(1-t)e^{2t} & -te^{2t} & e^{2t} \\ te^{2t} & (1+t)e^{2t} & 0\end{array}\right] \to \left[\begin{array}{cc|c}1-t & -t & 1 \\ t & 1+t & 0\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 1 & 1 \\ t & 1+t & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 1 \\ 0 & 1 & -t\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & t+1 \\ 0 & 1 & -t\end{array}\right]$$

$$c_1'(t) = t+1 \qquad c_1(t) = \tfrac{1}{2}t^2 + t$$
$$c_2'(t) = -t \qquad c_2(t) = -\tfrac{1}{2}t^2$$

$$\vec{p}(t) = \begin{bmatrix}(\frac{1}{2}t^2+t)(1-t) \\ (\frac{1}{2}t^2+t)\, t\end{bmatrix} e^{2t} + \begin{bmatrix}-\frac{1}{2}t^2(-t) \\ -\frac{1}{2}t^2(1+t)\end{bmatrix} e^{2t}$$

$$= \begin{bmatrix}\frac{1}{2}t^2 + t - \cancel{\frac{1}{2}t^3} - t^2 + \cancel{\frac{1}{2}t^3} \\ \cancel{\frac{1}{2}t^3} + t^2 - \frac{1}{2}t^2 - \cancel{\frac{1}{2}t^3}\end{bmatrix} e^{2t} = \begin{bmatrix}-\frac{1}{2}t^2 + t \\ \frac{1}{2}t^2\end{bmatrix} e^{2t}$$

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix}1-t \\ t\end{bmatrix} e^{2t} + c_2\begin{bmatrix}-t \\ 1+t\end{bmatrix} e^{2t} + \begin{bmatrix}t - \frac{1}{2}t^2 \\ \frac{1}{2}t^2\end{bmatrix} e^{2t}}$$

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#5 $$f(t) = \begin{cases} t^2 & \text{for } t < 1 \\ t^2 - 1 & 1 \le t < 2 \\ t^2 - 2 & t \ge 2 \end{cases}$$

(a)
$$= t^2 - u_1(t) - u_2(t)$$

(b) $$\mathcal{L}[f(t)](s) = \mathcal{L}[t^2](s) - \mathcal{L}[u_1(t)](s) - \mathcal{L}[u_2(t)](s)$$
$$= \frac{2}{s^3} - \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

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#6

(a) $\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+3}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+3)}\right](t)$

$\frac{A}{s+1}+\frac{B}{s+3} = \frac{1}{(s+1)(s+3)}$ $= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t)$

+4 $A(s+3)+B(s+1) = 1$ $= \boxed{\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}}$

$s=-1: \quad 2A = 1 \Rightarrow A = \frac{1}{2}$

$s=-3: \quad -2B = 1 \Rightarrow B = -\frac{1}{2}$

(b) $\mathcal{L}\left[t^2e^{3t}\right](s) = (-1)^2\frac{d^2}{ds^2}\mathcal{L}[e^{3t}](s) = \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \boxed{\frac{2}{(s-3)^2}}$

OR

+4 $\mathcal{L}\left[t^2e^{3t}\right](s) = \mathcal{L}[t^2](s-3) = \boxed{\frac{2}{(s-3)^3}}$

(c) $(D+1)x = e^{-t}, \quad x(0) = 1$

$\mathcal{L}[Dx+x](s) = \mathcal{L}[e^{-t}](s)$

$-x(0) + (s+1)\mathcal{L}[x](s) = \frac{1}{s+1}$

$\mathcal{L}[x](s) = \frac{1}{(s+1)^2} + \frac{1}{s+1}$

$x(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)$

$= e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) + e^{-t}$

$= te^{-t} + e^{-t}$

+4 $\boxed{x(t) = (t+1)e^{-t}}$

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#7. If, for some $j < k$, $v_j = v_k$ in the set of vectors $v_1, v_2, \ldots, v_n$,

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then $0v_1 + \cdots + 0v_{j-1} + v_j + 0v_{j+1} + \cdots$

$$+ 0v_{k-1} - v_k + 0v_{k+1} + \cdots + 0v_n = 0$$

is a linear combination with constants not all equal to zero.

$\therefore$ $v_1, \ldots, v_n$ are linearly dependent.

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#8 $(D^2+1)x = 0$ or $D^2x = -x$

(a) Let $x_1 = x$ and $x_2 = Dx$

Then $Dx_1 = x_2$
$Dx_2 = -x_1$

and $D\vec{x} = A\vec{x}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

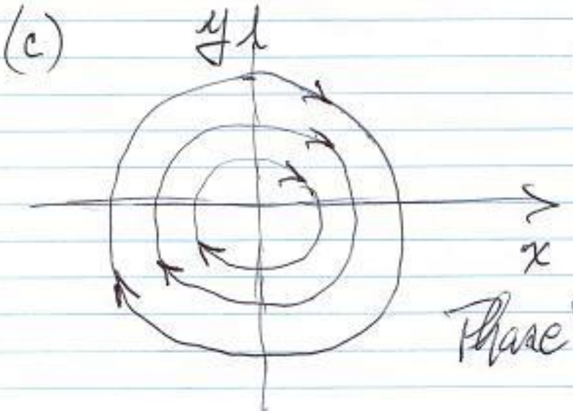
(b) $p(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$ $\quad \lambda = \pm i$

$\underline{\lambda = -i}$ $\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right]$ $\quad \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\begin{bmatrix} i \\ 1 \end{bmatrix}(\cos t - i\sin t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$$

(c)



At $(0,1)$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{dx}{dt} = 1$

Phase Portrait

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#9 $\frac{dx}{dt} = x^3 + y$

(12) $\frac{dy}{dt} = -x + y^3$

$\frac{dx}{dt} = \frac{dy}{dt} = 0$ only at $x = y = 0$

(a) Equilibrium point $(0, 0)$

(b) $E(x,y) = -x^2 - y^2$

$\frac{\partial E}{\partial x} = -2x \qquad \frac{\partial E}{\partial y} = -2y$

$\frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} = -2x^4 - 2xy + 2xy - 2y^4$

$= -2(x^4 + y^4) < 0$ on

all integral curves that are not equilibrium solutions

$\Rightarrow$ Lyapunov function

(c) $\frac{\partial^2 E}{\partial x^2} = -2 \qquad \frac{\partial^2 E}{\partial y^2} = -2 \qquad \frac{\partial^2 E}{\partial x \partial y} = 0$

$\Delta(x,y) = 4 > 0$ everywhere

$\Delta(0,0) > 0$ in particular

$\frac{\partial^2 E}{\partial x^2} = -2 < 0$ concave down

maximum at $(0,0) \Rightarrow$ repeller

(d) The total derivative of the Lyapunov function $E(x,y) = -x^2 - y^2$ is strictly negative on every integral curve. Hence, for any $t_2 > t_1$, we cannot have $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$, i.e., there are no closed integral curves.