Mathematics 38 | Tufts UNIVERSITY | **Differential Equations**
Final Exam | | **December 15, 2010 8:30 - 10:30 AM**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (36 points, 6 each) **These questions have no partial credit.**

a. Check for independence.

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}4\\5\\6\\7\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

b. Use the definition to compute $\mathscr{L}\left[e^{-t}\right]$.

c. Find $\mathscr{L}\left[te^{2t}\sin 3t\right]$.

d. Evaluate $1 * t$.

e. Find all solutions of the form $x = e^{at}$ or $x = t^a$ for the equation

$$(t^2D^2 - tD)x = 0$$

f. Solve the nonhomogeneous equation

$$(t^2D^2 - tD)x = t^{-1} \qquad t > 0$$

The questions below have partial credit.

2. (5 points) Solve $tx' - x = t^3 \qquad x(1) = 0$

3. (8 points)

a. Show that for any $b \geq 0$

$$x(t) = \begin{cases} 0 & t \leq b \\ (t-b)^5 & t > b \end{cases}$$

is a solution of

$$(*) \quad \frac{dx}{dt} = 5x^{4/5} \qquad x(0) = 0$$

b. Does $(*)$ have a unique solution?

c. Does this fact contradict the existence and uniqueness theorem? Explain why or why not.

Examination continues on other side

4. (5 points) Solve

$$\begin{pmatrix} 1 & 2 & 1 & -1 & -1 \\ 2 & 2 & 2 & -3 & -2 \\ -1 & 0 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

5. (10 points) Solve

$$D\vec{x} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{x}$$

6. (8 points) Solve

$$D\vec{x} = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -3 \\ 0 & 0 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{2t} \\ 3 \\ e^{2t} \end{pmatrix}$$

7. (8 points) Find the inverse Laplace transform of

a. $\dfrac{s+1}{s^2+6s+9}$

b. $\dfrac{1}{s^3+6s^2+9s}$

8. (10 points) Solve

$$(D^2+1)x = \begin{cases} \sin t & t \leq \pi \\ 0 & t \geq \pi \end{cases} \qquad x(0) = x'(0) = 0$$

9. (10 points) Consider the system

$$\text{(S)} \qquad \begin{aligned} \frac{dx}{dt} &= 10x - 6y \\ \frac{dy}{dt} &= -6x + 10y \end{aligned}$$

a. Show that $E = -5x^2 + 6xy - 5y^2$ is a Lyapunov function for (S).

b. Determine whether (S) has closed integral curves.

END OF EXAMINATION