Mathematics 38 **Differential Equations**
Final Exam **December 15, 2010 8:30 - 10:30 AM**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away. **You must show all your work** in the blue book in order to receive full credit. Please box your answers and **cross out any work you do not want graded.** Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (36 points, 6 each) **These questions have no partial credit.**

a. Check for independence.

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}4\\5\\6\\7\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Solution: Not linearly independent. Five 4-vectors cannot be independent. **Or:** $\begin{pmatrix}4\\5\\6\\7\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} - 3\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

b. Use the definition to compute $\mathscr{L}\left[e^{-t}\right]$.

Solution: $\mathscr{L}\left[e^{-t}\right] = \int_0^\infty e^{-t}e^{-st}\,dt = \int_0^\infty e^{-t(s+1)}\,dt = \frac{-1}{s+1}\lim_{t\to\infty}\left(e^{-t(s+1)} - 1\right) = \frac{1}{s+1}$.

c. Find $\mathscr{L}\left[te^{2t}\sin 3t\right]$.

Solution:
$\mathscr{L}\left[te^{2t}\sin 3t\right] = \mathscr{L}\left[t\sin 3t\right]\big|_{s-2} = \left(\frac{-d}{ds}\mathscr{L}\left[\sin 3t\right]\right)_{s-2} = \left(\frac{-d}{ds}\frac{3}{s^2+9}\right)_{s-2} = \left(\frac{6s}{s^2+9}\right)_{s-2} = \frac{6s-12}{(s^2-4s+13)^2}$.

d. Evaluate $1 * t$.

Solution: $1 * t = \int_0^t u\,du = \frac{1}{2}t^2$.

e. Find all solutions of the form $x = e^{at}$ or $x = t^a$ for the equation

$$(t^2D^2 - tD)x = 0$$

Solution: For $x(t) = e^{at}$, we have $(t^2D^2 - tD)x = (a(a-1)t - a)te^{at} = 0$ for all t only if $a = 0$. For $x(t) = t^a$, we have $(t^2D^2 - tD)x = t^2a(a-1)t^{a-2} - tat^{t-1} = a(a-2)t^a = 0$ for all t when $a = 0$ and $a = 2$, so the sought solutions are $x(t) = 1$ and $x(t) = t^2$.

f. Solve the nonhomogeneous equation

$$(t^2D^2 - tD)x = t^{-1} \qquad t > 0$$

Solution: First solve the homogeneous equation associated with $(D^2 - t^{-1}D)x = t^{-3}$: From **e.** we have $h_1(t) = 1$ and $h_2(t) = t^2$. Now one can try to proceed by inspection (try a solution of the form $c \cdot t^a$ with $c \neq 0$) or use variation of parameters: The system

$$k_1'1 + k_2't^2 = 0$$
$$k_1'0 + k_2'2t = t^{-3}$$

has solutions $k_1'(t) = -\frac{1}{2}t^{-2}$ and $k_2'(t) = \frac{1}{2}t^{-4}$ so $k_1(t) = \frac{1}{2}t^{-1}$ and $k_2(t) = -\frac{1}{6}t^{-3}$.
A particular solution is

$$p(t) = k_1(t)h_1(t) + k_2(t)h_2(t) = \frac{1}{2}t^{-1} - \frac{1}{6}t^{-1} = \frac{1}{3}t^{-1} = \frac{1}{3t}$$

so the general solution is

$$x(t) = c_1 + c_2t^2 + \frac{1}{3t}.$$

Note that we have been able to divide by t in several places since $t > 0$.

The questions below have partial credit.

2. (5 points) Solve $tx' - x = t^3 \qquad x(1) = 0$

Solution: The homogeneous equation can be solved by inspection or with separation of variables ($\int \frac{1}{x}\,dx = \int \frac{1}{t}\,dt$):

$$x(t) = ct + p(t),$$

and by inspection (try a solution of the form $c \cdot t^a$ with $c \neq 0$) or variation of parameters ($c'(t) \cdot t = t^3/t$, so $c(t) = \int t\;dt = \frac{1}{2}t^2$), a particular solution is $p(t) = \frac{1}{2}t^3$. To match the initial condition $0 = x(1) = c + \frac{1}{2}$, the solution is $x(t) = \frac{1}{2}(t^3 - t)$.

3. (8 points)

a. Show that for any $b \geq 0$

$$x(t) = \begin{cases} 0 & t \leq b \\ (t-b)^5 & t > b \end{cases}$$

is a solution of

$$(*) \quad \frac{dx}{dt} = 5x^{4/5} \qquad x(0) = 0$$

Solution: $x(0) = 0$ since $0 \leq b$. The main point is to "plug it in": $\frac{d}{dt}x(t) = \begin{cases} 0 & t \leq b \\ 5(t-b)^4 & t > b \end{cases} = 5x(t)^{4/5}$.

b. Does $(*)$ have a unique solution?

Solution: No. We just showed that $x(t) = \begin{cases} 0 & t \leq b \\ (t-b)^5 & t > b \end{cases}$ is a solution for *any* $b \geq 0$. That's infinitely many right there. Moreover, $x(t) = 0$ for all t is yet another solution of $(*)$.

c. Does this fact contradict the existence and uniqueness theorem? Explain why or why not.

Solution: No. $\frac{d}{dx}5x^{4/5} = 4x^{-1/5}$ is not continuous at $x = 0$, so the existence and uniqueness theorem does not apply when $x(0) = 0$.

4. (5 points) Solve

$$\begin{pmatrix} 1 & 2 & 1 & -1 & -1 \\ 2 & 2 & 2 & -3 & -2 \\ -1 & 0 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Solution: The system row-reduces to $\begin{pmatrix} 1 & 2 & 1 & -1 & -1 & | & 2 \\ 0 & -2 & 0 & 5 & 4 & | & -3 \\ 0 & 0 & 0 & 0 & 0 & | & -1 \end{pmatrix}$, which has no solution.

5. (10 points) Solve

$$D\vec{x} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{x}$$

Solution: By inspection or computation, the eigenvalues are $\lambda = 0, 2, 4$, with eigenvectors $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, so the general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \vec{x}(t) = \begin{pmatrix} 2(c_1 + c_3 e^{4t}) \\ c_2 e^{2t} \\ -c_1 + c_3 e^{4t} \end{pmatrix}.$$

6. (8 points) Solve

$$D\vec{x} = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -3 \\ 0 & 0 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{2t} \\ 3 \\ e^{2t} \end{pmatrix}$$

Solution: The eigenvalues are $\lambda = 2, \pm 3i$. For $\lambda = 2$ an eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and for $\lambda = 3i$ an eigenvector is $\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$.
From the "solution" $e^{3it}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = (\cos 3t + i \sin 3t)\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 3t + i \sin 3t \\ \sin 3t - i \cos 3t \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 3t \\ \sin 3t \\ 0 \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{pmatrix}$
we extract the real and imaginary parts to form the real solutions

$$\vec{h}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{h}_2(t) = \begin{pmatrix} \cos 3t \\ \sin 3t \\ 0 \end{pmatrix}, \quad \vec{h}_3(t) = \begin{pmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{pmatrix}$$

of the associated homogeneous system. To find a particular solution ($\vec{p}(t) = k_1(t)\vec{h}_1(t) + k_2(t)\vec{h}_2(t) + k_3(t)\vec{h}_3(t)$), use variation of parameters and solve the system

$$\left(\begin{array}{ccc|c} e^{2t} & \cos 3t & \sin 3t & e^{2t} \\ 0 & \sin 3t & -\cos 3t & 3 \\ e^{2t} & 0 & 0 & e^{2t} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 0 & \cos 3t & \sin 3t & 0 \\ 0 & \sin 3t & -\cos 3t & 3 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

to obtain $k_1'(t) = 1$ and, via Cramer's Rule or inspection, $k_2'(t) = 3\sin 3t$ and $k_3'(t) = -3\cos 3t$. Then $k_1(t) = t$, $k_2(t) = -\cos 3t$, and $k_3(t) = -\sin 3t$, giving the particular solution

$$\vec{p}(t) = te^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\cos^2 3t \\ -\cos 3t \sin 3t \\ 0 \end{pmatrix} + \begin{pmatrix} -\sin^2 3t \\ \cos 3t \sin 3t \\ 0 \end{pmatrix} = \begin{pmatrix} te^{2t} - 1 \\ 0 \\ te^{2t} \end{pmatrix}$$

and the general solution

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \cos 3t \\ \sin 3t \\ 0 \end{pmatrix} + c_3\begin{pmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{pmatrix} + \begin{pmatrix} te^{2t} - 1 \\ 0 \\ te^{2t} \end{pmatrix}.$$

7. (8 points) Find the inverse Laplace transform of

a. $\dfrac{s+1}{s^2+6s+9}$

Solution: $\mathscr{L}^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = \mathscr{L}^{-1}\left[\dfrac{s+1}{(s+3)^2}\right] = e^{-3t}\mathscr{L}^{-1}\left[\dfrac{s-2}{s^2}\right] = e^{-3t}\mathscr{L}^{-1}\left[\dfrac{1}{s} - \dfrac{2}{s^2}\right] = e^{-3t}(1-2t).$

b. $\dfrac{1}{s^3+6s^2+9s}$

Solution:

$$\frac{1}{s^3+6s^2+9s} = \frac{1}{s(s+3)^2} = \frac{1}{3}\frac{(s+3)-s}{s(s+3)^2} = \frac{1}{3}\left[\frac{1}{s(s+3)} - \frac{1}{(s+3)^2}\right] = \frac{1}{3}\left[\frac{1}{3}\frac{(s+3)-s}{s(s+3)} - \frac{1}{(s+3)^2}\right]$$
$$= \frac{1/9}{s} - \frac{1/9}{s+3} - \frac{1/3}{(s+3)^2},$$

so

$$\mathscr{L}^{-1}\left[\frac{1}{s^3+6s^s+9s}\right] = \frac{1}{9}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{9}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{(s+3)^2}\right] = \frac{1}{9} - \frac{1}{9}e^{-3t} - \frac{1}{3}te^{-3t},$$

where the third term uses the first shift formula.

8. (10 points) Solve

$$(D^2+1)x = \begin{cases} \sin t & t \le \pi \\ 0 & t \ge \pi \end{cases} \qquad\qquad x(0) = x'(0) = 0$$

Solution: Using the unit step function, the equation becomes $(D^2+1)x = \sin t - u_\pi(t)\sin t$. Applying the Laplace transform and using that $x(0) = x'(0) = 0$ gives

$$\begin{aligned}(s^2+1)\mathscr{L}[x] &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}[\sin(t+\pi)] \\ &= \frac{1}{s^2+1} + e^{-\pi s}\mathscr{L}[\sin t] \\ &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1},\end{aligned}$$

so $\mathscr{L}[x] = \dfrac{1}{(s^2+1)^2} + e^{-\pi s}\dfrac{1}{(s^2+1)^2}$ and

$$\begin{aligned}x(t) &= \sin t * \sin t + u_\pi(t)\,(\sin t * \sin t)_{t-\pi} \\ &= \frac{1}{2}\sin t - \frac{t}{2}\cos t + u_\pi(t)\left(\frac{1}{2}\sin(t-\pi) - \frac{t-\pi}{2}\cos(t-\pi)\right) \\ &= \frac{1}{2}\left[\sin t - t\cos t + u_\pi(t)\,(-\sin t + (t-\pi)\cos t)\right] \\ &= \frac{1}{2}\cdot\begin{cases} \sin t - t\cos t & t \le \pi \\ \quad -\pi\cos t & t \ge \pi.\end{cases}\end{aligned}$$

Note that this is continuous and these terms are as expected from undetermined coefficients.

9. (10 points) Consider the system

$$\text{(S)} \qquad \begin{aligned}\frac{dx}{dt} &= 10x - 6y \\ \frac{dy}{dt} &= -6x + 10y\end{aligned}$$

a. Show that $E = -5x^2 + 6xy - 5y^2$ is a Lyapunov function for (S).

Solution: $\dfrac{d}{dt}E(x(t), y(t)) = (-10x+6y)(10x-6y) + (6x-10y)(-6x+10y) = -(10x-6y)^2 - (6x-10y)^2 < 0$ except at the origin (the sole equilibrium of (S) and the sole critical point of E), so this is a Lyapunov function.

b. Determine whether (S) has closed integral curves.

Solution: It does not, because it has a Lyapunov function. **Or:** because $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial x} = 10 + 10 = 20 > 0.$

END OF EXAMINATION