Mathematics 38 — **Differential Equations**
Final Examination — **May 10, 8:30–10:30**

Tufts UNIVERSITY

No calculators, notes, or books are allowed. Please make sure all electronic devices are turned off and out of sight. Show all work and cross out work you do not want graded!
Remember to sign your blue book.
With your signature you are pledging that you have neither given nor received assistance on this exam. Good luck!

Part I: No partial credit

Please write the answers to problems 1–6 on the cover of the blue book to the right of the corresponding box.

1. (4 points) Are the vectors $\begin{pmatrix}1\\0\\-1\\5\end{pmatrix}$, $\begin{pmatrix}3\\1\\2\\0\end{pmatrix}$, $\begin{pmatrix}5\\-7\\2\\1\end{pmatrix}$ linearly independent?

Solution: Yes: $\begin{pmatrix}1&3&5\\0&1&-7\\-1&2&2\\5&0&1\end{pmatrix} \xrightarrow{R_3\to R_3+R_1+R_2} \begin{pmatrix}1&3&5\\0&1&-7\\0&6&0\\5&0&1\end{pmatrix} \xrightarrow{R_2\to R_2-R_3/6} \begin{pmatrix}1&3&5\\0&0&-7\\0&6&0\\5&0&1\end{pmatrix}$, which has no free variables.

2. (4 points) Are the vectors $\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\begin{pmatrix}3\\-1\\2\end{pmatrix}$, $\begin{pmatrix}5\\2\\1\end{pmatrix}$ linearly independent?

Solution: Yes: $\begin{pmatrix}1&3&5\\0&-1&2\\1&2&1\end{pmatrix} \xrightarrow{R_1\to R_1+R_2-R_3} \begin{pmatrix}0&0&6\\0&-1&2\\1&2&1\end{pmatrix}$, which has no free variables.

3. (4 points) Are the vectors $\begin{pmatrix}1\\5\\3\end{pmatrix}$, $\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\begin{pmatrix}7\\-2\\1\end{pmatrix}$, $\begin{pmatrix}4\\-3\\-2\end{pmatrix}$ linearly independent?

Solution: Obviously not (4 3-vectors).

4. Consider the differential equation $t\dfrac{dx}{dt} = x-2.$

a. (2 points) Is this equation linear?

Solution: Yes.

b. (2 points) What is the largest interval containing $t=1$ on which the equation is normal?

Solution: $(0,\infty)$.

c. (2 points) Does the existence and uniquenes theorem apply for $x(0)=2$? No, the differential equation is not normal for $t=1$.

d. (2 points) How many solutions of this differential equation satisfy $x(0)=2$)?
(None? Exactly one? Infinitely many?)

Solution: Infinitely many: $x(t)=At+2$ for every A.

e. (2 points) How many solutions of the equation satisfy $x(0)=0$?

Solution: Obviously none: For $t=0$ the differential equation reads $0x'=x-2$ and hence implies $x=2$ rather than $x=0$.

5. (5 points) Find the general solution of $(D^2+2)^2D(D^2-1)x=0.$

Solution: $c_1\sin\sqrt{2}t+c_2\cos\sqrt{2}t+c_3t\sin\sqrt{2}t+c_4t\cos\sqrt{2}t+c_5+c_6e^t+c_7e^{-t}.$

6. (6 points) Find the annihilator of

a. t^4,

Solution: D^5.

b. $te^{2t}+\sin t$.

Solution: $(D-2)^2(D^2+1)$.

Part II: Show all work and/or give correct reasons for your answers.

7. (11 points) The general solution of $((t-1)D^2 - tD + 1)x = 0$ for $t > 1$ is $h(t) = c_1 t + c_2 e^t$. You do not have to verify this.

Find the solution of $((t-1)D^2 - tD + 1)x = (t-1)^2 e^t$ that satisfies $x(2) = 1$, $x'(2) = e^2$.

Solution: $\begin{pmatrix} t & e^t & \mid & 0 \\ 1 & e^t & \mid & (t-1)e^t \end{pmatrix} \rightarrow \begin{pmatrix} t & e^t & \mid & 0 \\ 1-t & 0 & \mid & (t-1)e^t \end{pmatrix}$ gives $c_1'(t) = -e^t$ and then $c_2'(t) = t$. This gives the general solution $c_1 t + c_2 e^t + (\frac{1}{2}t^2 - t)e^t$. $1 = x(2) = 2c_1 + c_2 e^2$ and $e^2 = x'(2) = c_1 + c_2 e^2 + e^2$ give (by Cramer's Rule or row reduction) $c_1 = 1$ and then $c_2 = -e^{-2}$, so

$$x(t) = t - e^{t-2} + (\frac{1}{2}t^2 - t)e^t.$$

8. (12 points)

a. Find the eigenvalues of $A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}$.

Solution:
$\det \begin{pmatrix} 1-\lambda & -2 & 2 \\ 0 & 1-\lambda & 1 \\ -1 & -2 & 4-\lambda \end{pmatrix} = (1-\lambda)[(1-\lambda)(4-\lambda)+2] - [-2-2(1-\lambda)] = (1-\lambda)(\lambda-2)(\lambda-3) - 2(\lambda-2) = -(\lambda-2)[(\lambda-3)(\lambda-1)+2] = -(\lambda-2)[(\lambda-2)^2+1]$ has roots $\lambda = 2, 2 \pm i$.

b. Find the corresponding eigenvectors. (An answer of $\vec{0}$ in **b.** or **c.** will forfeit all credit.)

Solution: $\lambda = 2 \curvearrowright \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\lambda = 2+i \curvearrowright \vec{v} = \begin{pmatrix} 2 \\ 1-i \\ 2 \end{pmatrix}$, $\lambda = 2-i \curvearrowright \vec{v} = \begin{pmatrix} 2 \\ 1+i \\ 2 \end{pmatrix}$.

c. Find the general solution of $D\vec{x} = A\vec{x}$.

Solution: $\vec{x}(t) = e^{2t}\Big[c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2\cos t \\ \cos t + \sin t \\ 2\cos t \end{pmatrix} + c_3 \begin{pmatrix} 2\sin t \\ \sin t - \cos t \\ 2\sin t \end{pmatrix}\Big]$.

9. (11 points) The eigenvalue of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ is 1 (triple). Find the general solution of $D\vec{x} = A\vec{x}$.

Solution: For a triple eigenvalue of a 3×3 matrix, *any* vector is a generalized eigenvector, so we choose the 3 canonical vectors and note that $(A-I)^2 = 0$. Then

$$\vec{x}(t) = c_1 e^t \Big[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Big] + c_2 e^t \Big[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Big]$$
$$+ c_3 e^t \Big[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Big] = c_1 \begin{pmatrix} 1 \\ t \\ t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1-t \\ -t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ t \\ 1+t \end{pmatrix}.$$

10. (11 points)

a. Write $f(t) = |(t-1)(t-2)|$ as an expression in step-function notation.

Solution: $f(t) = (t-1)(t-2) - 2u_1(t)(t-1)(t-2) + 2u_2(t)(t-1)(t-2)$.

b. Compute the Laplace transform of $f(t)$. (Remember to give reasons!)

Solution: By the second shift formula $\mathscr{L}[(t-1)(t-2) - 2u_1(t)(t-1)(t-2) + 2u_2(t)(t-1)(t-2)] = \mathscr{L}[t^2 - 3t + 2] - 2e^{-s}\mathscr{L}[t(t-1)] + 2e^{-2s}\mathscr{L}[(t+1)t] = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s} - 2e^{-s}(\frac{2}{s^3} - \frac{1}{s^2}) + 2e^{-2s}(\frac{2}{s^3} + \frac{1}{s^2}))$.

11. (10 points) Compute $\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$.

Solution: Partial fractions give

$$\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{s\big((s^2+4)-(s^2+1)\big)}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right] = \frac{1}{3}(\cos t - \cos 2t).$$

12. Consider the system

$$\frac{dx}{dt} = 2y - x - 3$$
$$\frac{dy}{dt} = 4 - 2x - y\ .$$

a. (2 points) Find the equilibria of the system.

Solution: $(1, 2)$ only.

b. (2 points) Decide whether for this system $E(x, y) = x^2 - 2x + y^2 - 4y$ is a constant of motion, a Lyapunov function or neither of these.

Solution:
$\frac{d}{dt}E = (2x-2)(2y-x-3) + (2y-4)(4-2x-y) = 2(x-1)(2(y-2)-(x-1)) + 2(y-2)(-2(x-1)-(y-2)) = -2[(x-1)^2 + (y-2)^2] < 0$ for $(x, y) \neq (1, 2)$, so E is a Lyapunov function..

c. (2 points) Classify all equilibria as attractors, repellers or neither of these.

Solution: $E(x, y) = x^2 - 2x + y^2 - 4y = (x-1)^2 + (y-2)^2 - 5$ obviously has a global minimum at $(1, 2)$, which is hence an attractor.

d. (3 points) Show that the system has no closed integral curves.

Solution: Several possible reasons: 1) There is a Lyapunov function. 2) $\frac{d}{dx}(2y-x-3) + \frac{d}{dy}(4-2x-y) = -2 < 0$ everywhere.

e. (3 points) Draw the phase portrait.

Solution: Note that the system is linear (and homogeneous after changing variables to $(x-1, y-2)$). Check that the eigenvalues are complex (since $(1, 2)$ is an attractor, they must have negative real part) to get inward spirals; by checking a test point (such as $(1, 1)$) see that they run clockwise.

END OF EXAMINATION