Mathematics 38 **Differential Equations**
Final Exam **December 16, 2011**

No calculators, books or notes are allowed on the exam. All electronic devices must be turned off and put away.**You must show all your work** in the blue book in order to receive full credit. Please box your answers and cross out any work you do not want graded. Make sure to sign your blue book. With your signature you are pledging that you have neither given nor received assistance on the exam. *Good luck!*

1. (5 points each, no partial credit) In parts **a., b., c.**, find all real values of α, if any, for which the given function is a solution of the given differential equation.

a. $x = \alpha$, $\qquad \dfrac{d^7x}{dt^7} + \dfrac{dx}{dt} - x = 7$

Solution: $x = \alpha \Rightarrow -x = 7 \Rightarrow \alpha = -7$.

b. $x = t^\alpha$, $t > 0$, $\qquad 16t^2xx'' + 3x^2 = 0$

Solution: $16t^2xx'' + 3x^2 = 16\alpha(\alpha-1)t^{2\alpha} + 3t^{2\alpha} = (16\alpha^2 - 16\alpha + 3)t^{2\alpha} = 0$ for all $t > 0$ if and only if $16\alpha^2 - 16\alpha + 3 = 0$ and hence $\alpha = 1/4, 3/4$.

c. $x = e^{\alpha t}$, $\qquad x'\sqrt{x} = 2e^{3t}$

Solution: $x'\sqrt{x} = \alpha e^{\alpha t}e^{\alpha t/2} = 2e^{3t}$ if $\alpha = 2$ and $3\alpha/2 = 3$, which is equivalent. So $\alpha = 2$.

d. Use the method of undetermined coefficients (the "annihilator method") to determine a simplified guess for a particular solution of

$$(D+3)^2(D^2+1)^4x = \sin t.$$

Obtain the simplest form possible, and leave your answer in terms of the undetermined coefficients.

Do not try to determine the coefficients!

Solution: $k_1t^4\cos t + k_2t^4\sin t$.

e. Consider the functions t^3 and $|t^3|$ defined on $-\infty < t < \infty$. Are they linearly independent? Give a succinct and definitive reason.

Solution: If $c_1t^3 + c_2|t^3| = 0$ for all t, take $t = 1$ to get $c_1 + c_2 = 0$ and $t = -1$ to get $-c_1 + c_2 = 0$; together these give $c_1 = 0$ and $c_2 = 0$, so these functions are linearly independent.

f. The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$ are linearly independent. Decide whether $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are linearly independent. Give a succinct and definitive reason.

Solution: If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = 0$ then also $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 + 0\vec{v}_5 = 0$. By linear independence of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$ this implies $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$ (and $0 = 0$).

g. Suppose $g(t)$ is a continuous function. Find a solution of the initial-value problem

$$x' = g(t)x, \quad x(1) = 0.$$

Simplify it as much as possible. (Hint: Think before applying standard techniques.)

Solution: $x(t) = 0$ for all t. By inspection this is a solution of the ODE (since $\dfrac{dx}{dt} = 0$), and it satisfies the initial condition. Since the theorem about existence and uniqueness applies, any different answer is wrong.

One could get this by separation of variables (but it is embarrassing to be caught doing so): We separate variables to get $\dfrac{dx}{x} = g(t)dt$ (if $x \neq 0$). Integrating (and forgetting about absolute

values), we get $\ln x = \int g(t)\,dt + C$, so $x = e^{C+\int g(t)\,dt} = Ae^{\int g(t)\,dt}$ for some constant A (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we can take $A = 0$, which gives $x(t) = 0$ for all t.

2. (5 points, no partial credit) Choose one answer. $\det\begin{pmatrix} 5 & 1 & \sin t & t^2+3 & 1 \\ 0 & 4 & e^t & e^t & 0 \\ 0 & 0 & 3 & \ln t & 8 \\ 0 & 0 & 0 & 2 & \sqrt{t} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$

a. 5, **b.** $2\sqrt{t}$, **c.** 120, **d.** $120 - \sin t - \frac{1}{4}e^t - \frac{1}{4}(t^2+3)e^t$.

e. None of the above.

Solution: c. because the matrix is triangular.

3. (10 points) Find the general solution of $\cos t\,\dfrac{dx}{dt} + x\sin t = \cos t \sin t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Solution: The associated homogeneous equation is $\cos t\dfrac{dx}{dt} + x\sin t = 0$.

Separation of variables: $\ln|x| = \displaystyle\int \frac{dx}{x} = \int \frac{-\sin t}{\cos t}\,dt = \ln\cos t + C$, so $h(t) = k\cos t$.

Variation of parameters: $k'(t)\cos t = \sin t$, so $k(t) = -\ln\cos t$, $\qquad x(t) = (c - \ln\cos t)\cos t$.

4. (15 points) Use the Laplace transform to solve

$$(D-2)(D-1)^2x = -2e^t \qquad \text{with} \quad x(0) = x'(0) = 0 \text{ and } x''(0) = 2.$$

Solution: By inspection (as in undetermined coefficients), no terms other than e^{2t}, e^t, te^t, t^2e^t are to be expected. Laplace-transform to $(s-2)(s-1)^2\mathscr{L}[x] - 2 = \dfrac{-2}{s-1}$, which becomes

$$\mathscr{L}[x] = \frac{2}{(s-2)(s-1)^2} - \frac{2}{(s-2)(s-1)^3} = \frac{2(s-1)-2}{(s-2)(s-1)^3} = \frac{2(s-2)}{(s-2)(s-1)^3} = \frac{2}{(s-1)^3},$$

so

$$x(t) = \mathscr{L}^{-1}[\frac{2}{(s-1)^3}] = t^2e^t.$$

Check: $(D-2)(D-1)^2t^2e^t = e^t(D-1)(D^2t^2 = -2e^t$, $x(0) = 0^2e^0 = 0$, $x'(0) = Dt^2e^t = e^t(D+1)t^2 = (2t+t^2)e^t = 0$ when $t = 0$, and $x''(0) = D^2t^2e^t = e^t(D+1)^2t^2 = e^t(D+1)(2t+t^2) = (2+4t+t^2)e^t = 2$ for $t = 0$.

5. (15 points) Find the general solution of $D\vec{x} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\vec{x}$.

Solution: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are (obviously!!) generalized eigenvectors for the triple eigenvalue -1. Each of these gives a solution

$$\begin{aligned} h_i(t) &= e^{-t}[I + t(A+I) + \frac{t^2}{2}(A+I)^2]\vec{v}_i \\ &= e^{-t}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\vec{v}_i = e^{-t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\vec{v}_i, \end{aligned}$$

so the sought general solution is $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}t\\1\\0\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}t^2/2\\t\\1\end{pmatrix}$.

6. (20 pts) Consider the system of differential equations

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x.$$

a. Check whether $x^2 + y^2$ is a Lyapunov function,

Solution: $\frac{d}{dx}(x^2 + y^2) = 2x \cdot (-y) + 2y \cdot x = 0$, not "$< 0$", so no.[1]

b. check whether $x^2 + y^2$ is a constant of motion,

Solution: Yes, by the preceding calculation.

c. find the equilibria. (Make absolutely sure to get them right; otherwise there will be very little partial credit.)

Solution: $(0, 0)$ only.

For each equilibrium

d. determine its stability,

Solution: The origin is stable because it is a minimum of a constant of motion **or** by solving this *linear* system.

e. decide whether it is an attractor, a repeller, or neither.

Solution: It is neither of these; same reasoning as for "stability."

f. Determine whether there are closed integral curves,

Solution: Yes, lots: By solving this *linear* system **or** because the origin is a minimum of a constant of motion **or** because $\frac{dr}{dt} = 0$ implies that integral curves lie on circles centered at $\vec{0}$.

g. draw a plausible phase portrait.

Solution: All circles centered at the origin with counterclockwise arrows (since $x' = -y$).

[1] Unless we take the book definition, which allows "≤ 0"—in which case we cannot draw some of the useful conclusions we otherwise get.

END OF EXAMINATION