

Midterm Exam 2 with Solutions

1. (15 points): Use the method of undetermined coefficients to find a particular solution of the equation

$$x'' + x' - 2x = -18te^{-2t}.$$

Solution: The characteristic equation corresponding to the homogeneous equation is $\lambda^2 + \lambda - 2 = 0$. The roots are $\lambda = 1$, $\lambda = -2$, and the general solution to the homogeneous equation is

$$H(t) = C_1e^t + C_2e^{-2t}.$$

The annihilator for the right side is $A(D) = (D+2)^2$. Therefore, any particular solution of the given equation is among the general solution to the equation $A(D)(D-1)(D+2)x = 0$, or $(D-1)(D+2)^3x = 0$. The general solution of this equation is

$$x = C_1e^t + (C_2 + C_3t + C_4t^2)e^{-2t}.$$

Hence, the simplified form for a particular solution has the form $x_p = (C_3t + C_4t^2)e^{-2t}$. Substituting this into equation one obtains $2C_4 - 3C_3 = 0$, and $-6C_4 = -18$, or $C_4 = 3$, $C_3 = 2$.

ANS: $x_p = (2t + 3t^2)e^{-2t}$.

2. (10 points): Solve the following equation by the method of variation of parameters

$$2x'' + 2x = \sec t.$$

No credit by any other method.

Solution: The functions $h_1(t) = \cos t$ and $h_2(t) = \sin t$ generate the general solution to the homogeneous equation $2x'' + 2x = 0$. Hence, we look for a particular solution in the form

$$x_p = c_1(t) \cos t + c_2(t) \sin t.$$

The system

$$\begin{aligned} \cos(t)c_1' + \sin(t)c_2' &= 0, \\ -\sin(t)c_1' + \cos(t)c_2' &= \frac{1}{2} \sec t, \end{aligned}$$

has a solution $c_1' = -\frac{1}{2} \tan t$ and $c_2' = \frac{1}{2}$, yielding $c_1(t) = \frac{1}{2} \ln |\cos t|$ and $c_2(t) = \frac{t}{2}$.

ANS: $x(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} \cos t \ln |\cos t| + \frac{t}{2} \sin t.$

3. (10 points): Find the Laplace transform of the following functions:

(a) $te^{-t} \sin 2t$

(b)
$$\begin{cases} 2, & \text{if } t < 2, \\ t, & \text{if } 2 \leq t < 3, \\ e^{t-3} + 3, & \text{if } 3 \leq t. \end{cases}$$

Solution:

(a) By the First Differentiation Formula, $\mathcal{L}[t \sin(2t)](s) = -\frac{d}{ds} \left[\frac{2}{s^2 + 4} \right] = \frac{4s}{(s^2 + 4)^2}$.

Then, by the First Shift Formula, $\mathcal{L}[e^{-t}t \sin(2t)](s) = \frac{4(s+1)}{[(s+1)^2+4]^2}$.

(b) $\mathcal{L}[2 + u_2(t) \cdot (t - 2) + u_3(t) \cdot [e^{t-3} + 3 - t]](s) = \frac{2}{s} + e^{-2s} \mathcal{L}[t](s) + e^{-3s} \mathcal{L}[e^t - t](s) = \frac{2}{s} + \frac{e^{-2s}}{s^2} + e^{-3s} \left(\frac{1}{s-1} - \frac{1}{s^2} \right)$ (Second Shift Formula).

4. (10 points): Find the inverse Laplace transform of the following functions:

- (a) $\frac{e^{-s}(1+s)}{2(s^2+1)}$
 (b) $\frac{s}{s^2-2s+2}$

Solution:

(a) $\mathcal{L}^{-1}\left[\frac{e^{-s}(1+s)}{2(s^2+1)}\right](t) = \frac{1}{2}u_1(t)(\cos(t-1) + \sin(t-1))$. (Second Shift Formula).

(b) $\mathcal{L}^{-1}\left[\frac{s}{s^2-2s+2}\right](t) = \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\right](t) = e^t(\cos t + \sin t)$.
 (First Shift Formula).

5. (10 points): Use the following steps to compute the convolution $t * (\sin t - e^t)$:

- (a) find the Laplace transform of $t * (\sin t - e^t)$;
 (b) find a partial fraction decomposition of the Laplace transform obtained in Part (a);
 (c) find the inverse Laplace transform of the partial fraction decomposition found in Part (b).

Solution:

(a) $\mathcal{L}[t * (\sin t - e^t)] = \mathcal{L}[t] \cdot \mathcal{L}[\sin t - e^t] = \frac{1}{s^2} \left(\frac{1}{1+s^2} - \frac{1}{s-1} \right)$

(b) $\frac{1}{s^2} \left(\frac{1}{1+s^2} - \frac{1}{s-1} \right) = \frac{1}{s^2(1+s^2)} - \frac{1}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1+s^2} - \frac{s-(s-1)}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1+s^2} - \frac{1}{s(s-1)} + \frac{1}{s^2}$
 $= \frac{2}{s^2} - \frac{1}{1+s^2} - \frac{1}{s-1} + \frac{1}{s}$.

(c) $t * (\sin t - e^t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{1}{1+s^2} - \frac{1}{s-1} + \frac{1}{s}\right] = 2t - \sin t - e^t + 1$.

ANS: $2t - \sin t - e^t + 1$.

6. (10 points): Compute $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right]$ using the convolution formula. No credit by any other method.

Solution: $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) * \mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) = t * e^{2t} = \int_0^t u e^{2(t-u)} du$
 $= e^{2t} \int_0^t u e^{-2u} du = e^{2t} \left(\frac{1}{4} - \frac{t}{2} e^{-2t} - \frac{1}{4} e^{-2t}\right).$

ANS: $\frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4}.$

7. (15 points): Solve the following initial value problem

$$\begin{aligned}x'' - 4x' + 5x &= f(t), \\x(0) &= 0, \quad x' = 1,\end{aligned}$$

$$\text{where } f(t) = \begin{cases} 5, & \text{if } t < 1 \\ 0, & \text{if } 1 \leq t. \end{cases}$$

Solution: Applying the Laplace transform to both sides of the equation we have

$$(s^2 - 4s + 5)\mathcal{L}[x] - 1 = \frac{5}{s} - e^{-s}\frac{5}{s},$$

so

$$\mathcal{L}[x] = \frac{1}{(s-2)^2 + 1} + \frac{5}{s[(s-2)^2 + 1]} - e^{-s}\frac{5}{s[(s-2)^2 + 1]}.$$

Now we have to compute inverse Laplace transforms of these three terms on the right. The first term has the inverse

$$\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2 + 1}\right] = e^{2t} \sin t.$$

Second term can be written in the partial fraction form

$$\frac{5}{s[(s-2)^2 + 1]} = \frac{1}{s} - \frac{s-2}{(s-2)^2 + 1} + \frac{2}{(s-2)^2 + 1},$$

therefore,

$$\mathcal{L}^{-1}\left[\frac{5}{s[(s-2)^2 + 1]}\right] = 1 - e^{2t} \cos t + 2e^{2t} \sin t.$$

Inverse of the third term due to the Second Shift Formula is

$$\mathcal{L}^{-1}\left[e^{-s}\frac{5}{s[(s-2)^2 + 1]}\right] = u_1(t) \cdot [1 - e^{2(t-1)} \cos(t-1) + 2e^{2(t-1)} \sin(t-1)].$$

Hence,

$$\mathbf{x}(t) = 1 - e^{2t} \cos t + 3e^{2t} \sin t - u_1(t) \cdot [1 - e^{2(t-1)} \cos(t-1) + 2e^{2(t-1)} \sin(t-1)].$$

8. (10 points): Reduce the following higher order linear differential equations to equivalent systems (DO NOT SOLVE):

(a) $x'' + (e^t + t)x' = \sec t$

(b) $(D + 1)^3 x = t$

Solution:

(a) Introducing $x_1 = x$ and $x_2 = x'$ we have

$$\begin{aligned}x_1' &= x_2, \\x_2' &= -(e^t + t)x_2 + \sec t.\end{aligned}$$

ANS: $Dx = Ax + E(t)$ with $A = \begin{pmatrix} 0 & 1 \\ 0 & -e^t - t \end{pmatrix}$ and $E(t) = \begin{pmatrix} 0 \\ \sec(t) \end{pmatrix}$

(b) Introducing $x_1 = x$, $x_2 = x'$ and $x_3 = x''$ we have

$$\begin{aligned}x_1' &= x_2, \\x_2' &= x_3, \\x_3' &= -x_1 - 3x_2 - 3x_3 + t.\end{aligned}$$

ANS: $Dx = Ax + E(t)$ with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$ and $E(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$.

9. (10 points): Determine whether the vector functions

$$h_1(t) = \begin{pmatrix} 3e^{4t} \\ e^{4t} \end{pmatrix} \quad \text{and} \quad h_2(t) = \begin{pmatrix} 3e^{-4t} \\ e^{-4t} \end{pmatrix}$$

generate the general solution of the system

$$\begin{aligned}x_1' &= 5x_1 - 3x_2, \\x_2' &= 3x_1 - 5x_2.\end{aligned}$$

Justify your answer.

Solution: (a) We can check that $h_2(t)$ does not satisfy the system,

(b) The Wronskian of these two vector functions is 0 for all t .

Either of these imply that the given vector functions **do not generate** the general solution.