

## Midterm Exam 3 with Solutions

1. (10 points): Check following sets of vectors for independence (NO PARTIAL CREDIT):

$$(a) \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$(b) \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \\ 8 \\ 10 \end{pmatrix}$$

**Solution:**

(a) Not linearly independent.

$$(b) 2 \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 6 \\ 8 \\ 8 \\ 10 \end{pmatrix} = \vec{0}. \text{ Hence, these vectors are not linearly independent.}$$

2. (15 points): (a) Find the eigenvalues of the matrix  $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

(b) Find eigenvectors for each eigenvalue of the matrix  $A$ .

**Solution:**

(a) Eigenvalues are 1, 2, 3 since the matrix is lower triangular.

(b) By inspection, the eigenvector corresponding to  $\lambda_3 = 3$  is  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . The eigenvector corresponding to  $\lambda_2 = 2$  is obtained by solving the system  $\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} v = 0$ , so

we get  $v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ . The eigenvector corresponding to  $\lambda_1 = 1$  is obtained by solving the system  $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} v = 0$ , and we have  $v = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ .

3. (10 points): The matrix  $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$  has eigenvectors  $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  and  $w = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ . Find the corresponding eigenvalues.

**Solution:** For  $v$  one has  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , and therefore, the eigenvalue is 2.

Similarly for  $w$  we have  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ , and the eigenvalue is -1.

4. (10 points): Use the row reduction method to solve the following system:

$$\begin{aligned}x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 1 \\-x_1 - x_3 + 2x_4 + x_5 &= 1.\end{aligned}$$

No credit for any other method.

**Solution:** In matrix notation

$$\begin{aligned}&\left(\begin{array}{ccccc|c}1 & 2 & 1 & -1 & -1 & -1 \\2 & 2 & 2 & -3 & -2 & 1 \\-1 & 0 & -1 & 2 & 1 & 1\end{array}\right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 + R_1} \left(\begin{array}{ccccc|c}1 & 2 & 1 & -1 & -1 & -1 \\0 & -2 & 0 & -1 & 0 & -3 \\0 & 2 & 0 & 1 & 0 & 3\end{array}\right) \\&\xrightarrow{R_2 \rightarrow (-\frac{1}{2})R_2} \left(\begin{array}{ccccc|c}1 & 2 & 1 & -1 & -1 & -1 \\0 & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{2} \\0 & 2 & 0 & 1 & 0 & 3\end{array}\right) \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left(\begin{array}{ccccc|c}1 & 2 & 1 & -1 & -1 & -1 \\0 & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{2} \\0 & 0 & 0 & 0 & 0 & 0\end{array}\right).\end{aligned}$$

Let  $x_3 = a$ ,  $x_4 = b$ ,  $x_5 = c$ , then  $x_1 = -a + 2b + c - 1$ ,  $x_2 = -\frac{b}{2} + \frac{3}{2}$ , so

$$x = \begin{pmatrix} -a + 2b + c - 1 \\ -\frac{b}{2} + \frac{3}{2} \\ a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ \frac{3}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5. (20 points): Solve the system of linear differential equations  $D\vec{x} = A\vec{x}$ , where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

**Solution:** Notice the block form. This is a decoupled pair of second order systems of differential equations. The characteristic polynomial of  $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  is  $(\lambda - 1)(\lambda - 3)$ , so

the eigenvalues are 1 and 3, 1 has eigenvector  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , 3 has eigenvector  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Next,

$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$  has eigenvalues  $\pm 2i$ . Take  $2i$  and  $\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} v = 0$ , so we get eigenvector

$\begin{pmatrix} -i \\ 1 \end{pmatrix}$ .

$$e^{2it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \cos 2t + i \sin 2t \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}.$$

So, the general solution is

$$c_1 e^t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ -\cos 2t \\ \sin 2t \end{pmatrix}$$

6. (15 points): The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

has a triple eigenvalue  $\lambda = 1$  (You do not have to verify this). Find the generalized eigenvectors of  $A$ .

$$\begin{aligned} \text{Solution: } (A - \lambda I)^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So, the generalized eigenvectors are  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

7. (20 points): Solve the following initial value problem:

$$D\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vec{x} \quad \vec{x}(0) = \begin{pmatrix} 10 \\ 20 \\ 10 \end{pmatrix}$$

**Solution:** The characteristic polynomial is  $\lambda^3 - 2\lambda^2$ , so  $\lambda_1 = 2$ , which gives eigenvector  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , and a double root  $\lambda_2 = 0$ . For  $\lambda_2 = 0$ , the corresponding eigenvectors, by

inspection are  $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ , and  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . (You will get the same vectors if you solve the system  $(A - \lambda I)^2 v = 0$ ).

So, the general solution is  $x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . Plugging in the initial values, we have  $c_1 = 15, c_2 = 5, c_3 = -5$ . Thus the solution is

$$x(t) = 15 \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + 5 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (-5) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$