Eigenvalues and characteristic polynomials

Calculation of eigenvalues for $n \times n$-matrices where $n \geq 3$ can be a very tedious task, since we have to find the roots of the characteristic polynomial.

In many cases, however, we can obtain directly the factorization (or partial factorization) of the characteristic polynomial by performing row and/or column operations inside:

$$\det(A - \lambda I).$$

In the following examples we will denote a row in a determinant by the letter R followed by an integer. Similarly columns are denoted by the letter C followed by an integer

Ex: R2 $\rightarrow$ row 2

C3 $\rightarrow$ column 3.

Examples:

1°. Find the eigenvalues for:

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -3 & 0 & 4 \\ -2 & -1 & 4 \end{pmatrix}$$

Replace C_1 by $C_1 + C_2 + C_3$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & -2 \\ -3 & -\lambda & 4 \\ -2 & -1 & 4-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} 1-\lambda & 1 & -2 \\ 1-\lambda & -\lambda & 4 \\ 1-\lambda & -1 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 1 & -\lambda & 4 \\ 1 & -1 & 4-\lambda \end{vmatrix} =$$

$$= (1-\lambda) \begin{vmatrix} 1 & \cancel{1} & \cancel{-2} \\ 0 & -(\lambda+1) & 6 \\ 0 & -2 & 6-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -(\lambda+1) & 6 \\ -2 & 6-\lambda \end{vmatrix} =$$

Replace:
$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= (1-\lambda)\left[-(\lambda+1)(6-\lambda) + 12\right] =$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6) = (1-\lambda)(\lambda-2)(\lambda-3)$$

$\Rightarrow$ Eigenvalues $1, 2, 3$

Example 2:

Find the eigenvalues for:

$$A = \begin{pmatrix} 0 & -2 & 2 \\ 1 & 3 & -2 \\ 2 & 4 & -3 \end{pmatrix}$$

Replace $R1 \to R1+R2$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -2 & 2 \\ 1 & (3-\lambda) & -2 \\ 2 & 4 & (-3-\lambda) \end{vmatrix} =$$

$$= \begin{vmatrix} 1-\lambda & 1-\lambda & 0 \\ 1 & (3-\lambda) & -2 \\ 2 & 4 & (-3-\lambda) \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ 1 & (3-\lambda) & -2 \\ 2 & 4 & (-3-\lambda) \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & (2-\lambda) & -2 \\ 2 & 2 & (-3-\lambda) \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} =$$

Replace:
$R_2 \to R_2 - R_1$

$$= (1-\lambda)\left[(2-\lambda)(-3-\lambda) + 4\right] =$$

$$= (1-\lambda)(\lambda^2+\lambda-2) = (1-\lambda)(\lambda-1)(\lambda+2) =$$

$$= -(\lambda-1)^2(\lambda+2)$$

Eigenvalues: $\lambda = 1, -2$