THE HARTMAN–GROBMAN THEOREM

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The Hartman–Grobman Theorem (see [3, page 353]) was proved by Philip Hartman in 1960 [5]. It had been announced by Grobman in 1959 [1], likely unbeknownst to Hartman, and Grobman published his proof in 1962 [2], likely without knowing of Hartman's work. (Grobman attributes the question to Nemycki and an earlier partial result to R. M. Minc (citing Nauč. Dokl. Vysš. Školy. Fiz.-Mat. Nauki 1 (1958)).

The point of these notes is to comment on refinements that make this result particularly useful for constructing phase portraits of systems of nonlinear differential equations.

1. Continuous changes of coordinates

1. **Theorem** ([1, 2, 5],[3, page 353])**.** *If an nth-order system of differential equations has an equilibrium $\vec{c}$ with linearization matrix A, and if A has no zero or pure imaginary eigenvalues, then the phase portrait for the system near the equilibrium is obtained from the phase portrait of the linearized system $D\vec{x} = A\vec{x}$ via a continuous change of coordinates.*

2. **Example** (A continuous change of coordinates)**.** Consider the change of variables $h(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} (x-y)^5 + 3x + y \\ -3(x-y)^5 + 3x + y \end{pmatrix}$. This is clearly continuous. It is also invertible. Indeed, $h^{-1}(\begin{pmatrix} x \\ y \end{pmatrix}) = \frac{1}{4}\begin{pmatrix} \sqrt[5]{(x-y)/4} + (3x+y)/4 \\ -3\sqrt[5]{(x-y)/4} + (3x+y)/4 \end{pmatrix}$.

3. **Remark.** Having this continuous change of coordinates is sufficient for classifying the equilibrium as stable or unstable and as an attractor, repeller, or neither of these. For drawing phase portraits it is not as helpful as one might think.

The Figures 4.11, 4.12 and 4.13 (with arrows ignored!) on pages 338–339 of [3] can be obtained from each other by a continuous change of coordinates. So can the two phase portraits in [3, Figure 4.16].

For instance, consider the continuous coordinate change in the example above. The solution curves of Example 4.2.2 in the book are given by $\begin{pmatrix} ae^t + be^{5t} \\ -3ae^t + be^{5t} \end{pmatrix}$; these are shown in Figure 4.11 of [3]. When we apply the change of variables, then these become

$$h(\begin{pmatrix} ae^t + be^{5t} \\ -3ae^t + be^{5t} \end{pmatrix}) = e^{5t}\begin{pmatrix} 4^5 a^5 + 4b \\ -3\cdot 4^5 a^5 + 4b \end{pmatrix}.$$

But these are of the form $e^{5t}\vec{v}$ and therefore move radially, just like the solutions shown in Figure 4.12 of [3], albeit with reversed arrows.

The point is that a *continuous* change of coordinates provides little information about the geometry of the phase portrait.

2. Smooth changes of coordinates for second-order systems

For second-order systems, Hartman proved a version of the Hartman–Grobman Theorem that has a much stronger conclusion:

4. **Theorem** ([6]). *For a* second-order *system with an equilibrium that satisfies the hypotheses of the Hartman–Grobman Theorem, the continuous change of coordinates can, in fact, be taken to be* continuously differentiable *and such that the derivative at the equilibrium is the identity.*

This is a significant improvement: no two of the figures referred to in Remark 3 are related by a continuously differentiable change of coordinates, so these kind of coordinate changes can keep the various types of equilibria apart much better. Moreover, taking any of these figures and rotating it produces a figure that is *not* related to the original one by a coordinate change whose derivative at the equilibrium is the identity.

What Theorem 4 is saying is that

> *The phase portrait of a second-order system of differential equations near a suitable equilibrium is obtained by taking the phase portrait of the linearization (at that equilibrium) and pasting a thumbnail of it at the equilibrium (without rotation).*

This gives much better geometric information than Theorem 1.

3. Higher-order systems

One might try to draw phase portraits of third-order systems of differential equations, and it may be of interest to describe equilibria of higher-order systems with greater geometric detail than Theorem 1 allows. The trouble is that Theorem 4 is not true for systems of order 3 and higher. Hartman gave counterexamples that showed this.

The good news is that in order to paste linearized thumbnails into a phase portrait requires less than what Theorem 4 says. It is enough for the change of variables to be differentiable at the equilibrium and for its differential at the equilibrium to be the identity. And, as luck would have it, this is actually always true for systems of *any* order. This was first proved in the third millennium:

5. **Theorem** ([4]). *For any nth-order system with an equilibrium that satisfies the hypotheses of the Hartman–Grobman Theorem, the continuous change of coordinates can, in fact, be taken to be differentiable at the equilibrium (as opposed to continuously differentiable everywhere) and such that the derivative at the equilibrium is the identity.*

This says that pasting thumbnails works in any dimension:

> *The phase portrait of any system of differential equations at an equilibrium that satisfies the hypotheses of the Hartman–Grobman Theorem is obtained by taking the phase portrait of the linearization (at that equilibrium) and pasting a thumbnail of it at the equilibrium (without rotation).*

4. Putting this to use

Example 4.3.1 in [3] shows how to apply the Hartman–Grobman Theorem in practice.

Step 1. Locate all equilibria.

Step 2. At each equilibrium, linearize the right-hand side of the system of differential equations.

Step 3. For each equilibrium at which the Hartman–Grobman Theorem applies, use the methods from [3, Section 4.2] to draw the phase portrait of the linearized system. In the example, these “thumbnails” are shown in Figure 4.20.

Step 4. Draw *one* picture in which each of the thumbnails from the previous step are reproduced, centered on the corresponding equilibrium of the original system. (Check that there are no mismatches in the directions of arrows that arise from different thumbnails.)

Step 5. In this picture attempt to connect the pieces of integral curves from the thumbnails to produce as much of a complete phase portrait as possible. For Example 4.3.1, the result of this is shown in Figure 4.21.

5. Further dicussion

[3, page 356] discusses the phase portrait obtained in the example. The point of drawing the phase portrait is to make it possible to describe the long-term behavior of solutions of the system of differential equations. Note that for Example 4.3.1 the simpler method of Section 4.1 gives some of the conclusions already. But Figure 4.21 also shows the curves (separatrices) between the regions of integral curves tending to $(0,1)$ and $(1,0)$, respectively. That at the equilibrium $(3/8,1/4)$ these are tangent to the contracting eigenvector of the linearization is explained by Theorem 5 above, which furthermore explains why the distinguished directions at the other equilibria match those of the eigenvectors for the linearizations.

Note that steps 1–5 above are also carried out in Examples 4.3.2 and 4.3.3 in [3].

References

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