COMPLEX NUMBERS, UNDETERMINED COEFFICIENTS, AND LAPLACE TRANSFORMS

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CONTENTS

The purpose of these notes is to introduce complex numbers and their use in solving ordinary differential equations. The main feature is that trigonometric functions can be omitted from the methods even when they arise in a given problem or its solution.

One one hand this approach is illustrated with the method of undetermined coefficients, where this approach offers a parallel development to the one in the text [1]. On the other hand, this is done with the Laplace transform, where the use of complex numbers replaces a great deal of machinery needed otherwise *and* at the same time covers situations that the methods in the text do not. The main effort here is to show how to produce partial-fractions decompositions using complex numbers.

The reason the method of undetermined coefficients is revisited here in the complex context is that fluency with this method is very helpful in using the Laplace transform method reliably.

The exercises included here are not for credit but will help you read this text actively.

1. COMPLEX NUMBERS, EULER'S FORMULA

We introduce the symbol i with the property

$$\boxed{i^2 = -1}$$

A complex number is an expression that can be written in the form $a + ib$ with real numbers a and b.

1.1. **Real and imaginary part, complex conjugate.** If a and b are *real* numbers, then a is called the *real part* of $a+ib$, and b is called the *imaginary part.* (Note that the imaginary part is a real number!)

The expression $a-ib$ is called the *complex conjugate* of $a+ib$. It is sometimes denoted by a bar:

$$\overline{a+ib}=a-ib.$$

Adding a complex number and its complex conjugate always gives a real number:

$$a+ib+a-ib=2a.$$

This is twice the real part. So, if we are given a complex number $z=a+ib$ in any form, we can express the real part as

$$\Re(z)=\text{real part of } z=\frac{z+\overline{z}}{2}.$$

The imaginary part can be expressed as (check!)

$$\Im(z)=\text{imaginary part of } z=\frac{z-\overline{z}}{2i}.$$

1. **Exercise.** Verify that $\overline{\overline{z}}=z$.

2. **Exercise.** Verify that any real number x satisfies $\overline{x}=x$.

3. **Exercise.** Verify that a complex number z satisfying $\overline{z}=z$ is a real number.

1.2. **Multiplying complex numbers.** To multiply two complex numbers just use $i^2=-1$ and group terms:

$$(a+ib)(c+id)=ac+aid+ibc+ibid=ac-bd+i(ad+bc).$$

Multiplying a complex number and its complex conjugate always gives a real number:

$$(a+ib)(a-ib)=a^2+b^2.$$

We call $\sqrt{a^2+b^2}$ the *absolute value* or *modulus* of $a+ib$:

$$\boxed{|a+ib|=\sqrt{a^2+b^2}}$$

4. **Exercise.** Verify that $|z|=\sqrt{z\overline{z}}$.

5. **Exercise.** Verify that $\overline{z_1}\,\overline{z_2}=\overline{z_1z_2}$.

6. **Exercise.** Verify that $\overline{\overline{z_1}z_2}=z_1\overline{z_2}$.

1.3. **Dividing complex numbers.** To divide two complex numbers and write the result as real part plus $i\times$imaginary part, multiply top and bottom of this fraction by the complex conjugate of the denominator:

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}.$$

1.4. **Factoring polynomials.** Factoring polynomials is no harder (or easier) when complex numbers are allowed *but* in this case all factors are linear. The reason is that factors $x-\alpha$ are now legal even when α is complex.

7. **Example.** The polynomial s^2+1 is irreducible over the real numbers, but we have

$$s^2+1 = (s-i)(s+i).$$

1.5. **Euler's formula.** Complex numbers are useful in our context because they give Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula is easy to remember. In case you are not sure whether to attach the i to the cos or to the sin, just plug in $\theta = 0$.

It's worthwhile recalling some power series to see why this is so. We have

$$\begin{aligned}
e^z &= 1+z+z^2/2+\ldots &&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\cos z &= 1-z^2/2+\ldots &&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\sin z &= z-z^3/6+\ldots &&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}
\end{aligned}$$

With $z = i\theta$ this gives

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!}$$

Here we have simply summed even and odd powers separately. The reason we did this is that we can rewrite

$$i^{2n} = (i^2)^n = (-1)^n \quad \text{and} \quad i^{2n+1} = i(i^2)^n = i(-1)^n.$$

If we pull out the leading is from the sum over odd powers of t, we find

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \cos\theta + i\sin\theta.$$

In summary, we will use the "forward" and "backward" Euler formulas

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}} \quad \boxed{\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{i}{2}(e^{-i\theta}-e^{i\theta})}$$

8. **Exercise.** Show that $e^{\overline{z}} = \overline{e^z}$.

9. **Exercise.** Verify that if λ is complex and $ae^{\lambda t} + be^{\overline{\lambda}t}$ is real for all t, then $b = \overline{a}$ via the following steps.

(1) Rewrite the given information using Exercise 2 (and Exercise 8).
(2) The deriviative of $ae^{\lambda t} + be^{\overline{\lambda}t}$ is also real—compute it and proceed as in the previous part.
(3) Take $t = 0$ in the expressions obtained in both previous parts.

1.6. **Hyperbolic functions.** While we think of t as a real variable in Euler's formula, one gets an interesting result when one plugs imaginary numbers into cos and sin:

$$\cos ix = \frac{e^{i(ix)}+e^{-i(ix)}}{2} = \frac{e^{-x}+e^{x}}{2} = \frac{e^{x}+e^{-x}}{2} =: \cosh x.$$

Here, $\cosh x$ is the hyperbolic cosine function defined by this last equality. Likewise,

$$\sin ix = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i} = i\frac{e^{x}-e^{-x}}{2} = i\sinh x.$$

1.7. **The trigonometric-identity machine.** Euler's formula produces many trigonometric identities by using the rules of exponents.

10. **Example.** Find formulas for $\cos(a+b)$ and for $\sin(a+b)$.

We start with the law of exponents $e^{i(a+b)} = e^{ia}e^{ib}$ and rewrite every exponential using Euler's formula. On the left we have

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b).$$

On the right we have

$$\begin{aligned} e^{ia}e^{ib} &= \big(\cos a + i\sin a\big)\big(\cos b + i\sin b\big) \\ &= \big(\cos a\cos b - \sin a\sin b\big) + i\big(\sin a\cos b + \cos a\sin b\big). \end{aligned}$$

Since these two things are the same, we get

$$\cos(a+b)+i\sin(a+b) = \big(\cos a\cos b-\sin a\sin b\big)+i\big(\sin a\cos b+\cos a\sin b\big).$$

Matching real and imaginary parts, we get

$$\cos(a+b) = \cos a\cos b-\sin a\sin b \text{ and } \sin(a+b) = \sin a\cos b+\cos a\sin b.$$

11. **Example.** To get double-angle formulas, take $a = b$ in the previous example. Note that $\cos 2a$ then simplifies to $\cos^2 a - \sin^2 a$.

12. **Example.** To obtain triple-angle formulas use the law $e^{3ia} = \left(e^{ia}\right)^3$ and rewrite every term using Euler's formula, multiply out and sort terms. (Or use Example 10 with $b = 2a$ and use double-angle formulas along the way.)

13. **Example.** Should you ever need a formula for $\cos(a + b + c)$, you can start with the law of exponents $e^{i(a+b+c)} = e^{ia}e^{ib}e^{ic}$ and rewrite every exponential using Euler's formula, then multiply out and sort terms.

2. Undetermined coefficients

This method works much like in [1, Section 2.7], but we use complex exponentials and complex factoring of polynomials where appropriate.

14. **Example** ([1, Example 2.7.5])**.** Solve

$$(D^2 - 4)x = 1 + 65e^t \cos 2t. \tag{N}$$

Since $D^2 - 4 = (D-2)(D+2)$, the general solution of the associated homogeneous equation

$$(D^2 - 4)x = 0 \tag{H}$$

is

$$x = H(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

The forcing term of (N) can be rewritten as $1 + \frac{65}{2}e^{(1+2i)t} + \frac{65}{2}e^{(1-2i)t}$ and is therefore annihilated by

$$D(D - (1 + 2i))(D - (1 - 2i)).$$

Therefore a particular solution $x = p(t)$ of (N) will also satisfy

$$D(D - (1 + 2i))(D - (1 - 2i))(D - 2)(D + 2)x = 0. \tag{H*}$$

This has characteristic polynomial $r(r - (1 + 2i))(r - (1 - 2i))(r - 2)(r + 2)$, so $p(t)$ has to be of the form

$$p(t) = k_1 + k_2 e^{(1+2i)t} + k_3 e^{(1-2i)t} + k_4 e^{2t} + k_5 e^{-2t}.$$

15. **Remark.** When we use this method later to predict what terms to expect in applying the Laplace transform method, then we can stop at this point.

We get a simplified guess by omitting the terms that solve (H):

$$p(t) = k_1 + k_2 e^{(1+2i)t} + k_3 e^{(1-2i)t}.$$

To determine the (as yet undetermined) coefficients k_1, k_2 and k_3 insert this simplified guess into (N), written with the forcing term on the left:

$$\begin{aligned}
1+\frac{65}{2}e^{(1+2i)t}+\frac{65}{2}e^{(1-2i)t} &= (D^2-4)p(t)\\
&= (D^2-4)[k_1+k_2e^{(1+2i)t}+k_3e^{(1-2i)t}]\\
&= (D^2-4)k_1+(D^2-4)k_2e^{(1+2i)t}\\
&\qquad\qquad +(D^2-4)k_3e^{(1-2i)t}\\
\text{(exponential shift)}\rightsquigarrow &= -4k_1+e^{(1+2i)t}([D+(1+2i)]^2-4)k_2\\
&\qquad +e^{(1-2i)t}([D+(1-2i)]^2-4)k_3\\
&= -4k_1+e^{(1+2i)t}((1+2i)^2-4)k_2\\
&\qquad +e^{(1-2i)t}((1-2i)^2-4)k_3
\end{aligned}$$

Since the functions 1, $e^{(1+2i)t}$ and $e^{(1-2i)t}$ are linearly independent, we can equate coefficients of like terms on left and right:

$$-4k_1=1,\qquad ((1+2i)^2-4)k_2=\frac{65}{2},\qquad ((1-2i)^2-4)k_3=\frac{65}{2}.$$

Thus, $k_1=-1/4$ and

$$k_2=\frac{65}{2((1+2i)^2-4)}=\frac{65}{2(1+4i-4-4)}=\frac{65}{2(4i-7)}=\frac{1}{2}(-4i-7)$$

and (either by a like computation or because it must be the complex conjugate[1])

$$k_3=\frac{1}{2}(4i-7).$$

This gives

$$\begin{aligned}
p(t)&=-\frac{1}{4}+\frac{1}{2}(-4i-7)e^{(1+2i)t}+\frac{1}{2}(4i-7)e^{(1-2i)t}\\
&=-\frac{1}{4}-\frac{4i}{2}e^t(e^{2it}-e^{-2it})-\frac{7}{2}e^t(e^{2it}+e^{-2it})\\
&=-\frac{1}{4}+4e^t\sin 2t-7e^t\cos 2t.
\end{aligned}$$

The general solution of (N) then is the sum of $H(t)$ and $p(t)$:

$$x=H(t)+p(t)=c_1e^{2t}+c_2e^{-2t}-\frac{1}{4}+4e^t\sin 2t-7e^t\cos 2t.$$

This completes the example. We note that when using trigonometric functions as in the text, one has to solve a 2×2 system of equations to

[1]Don't use the "must be the complex conjugate"-shortcut if you don't know why it's true!

find k_2 and k_3, while with complex exponentials one instead computes the reciprocal of a complex number.

3. Laplace transforms

The definition of the Laplace transform in the textbook [1, page 412] (see also [1, Example 5.2.1] and [1, Example 5.2.4]) gives

$$
\begin{aligned}
\mathscr{L}[t^n e^{\lambda t}] &= \int_0^\infty e^{-st} t^n e^{\lambda t}\, dt \\
&= \int_0^\infty t^n e^{-(s-\lambda)t}\, dt \\
\left(\substack{\text{integration by parts if } n \geq 1: \\ u=t^n,\ dv=e^{-(s-\lambda)t}\, dt}\right) \rightsquigarrow &= \begin{cases} \dfrac{1}{s-\lambda} & \text{if } n=0 \\ 0 + \dfrac{n}{s-\lambda} \displaystyle\int_0^\infty t^{n-1} e^{-(s-\lambda)t}\, dt & \text{if } n \geq 1 \end{cases} \\
&= \begin{cases} \dfrac{1}{s-\lambda} & \text{if } n=0 \\ \\ \dfrac{n}{s-\lambda} \mathscr{L}[t^{n-1} e^{\lambda t}] & \text{if } n \geq 1. \end{cases}
\end{aligned}
$$

Applying this recursively, we find

$$
\boxed{\mathscr{L}[t^n e^{\lambda t}] = \frac{n!}{(s-\lambda)^{n+1}} \qquad \mathscr{L}^{-1}\left[\frac{1}{(s-\lambda)^n}\right] = \frac{1}{(n-1)!} t^{n-1} e^{\lambda t}} \tag{1}
$$

This will go quite far when combined with partial fractions. Note that for $n = 0$ this gives the Laplace transform of an exponential function and for $a = 0$ it gives the Laplace transform of a power of t.

4. Partial fractions

Partial-fractions decompositions are *easier* with complex numbers, because irreducible quadratics are no longer a difficulty: everything reduces to linear factors. One just has to patiently work the partial-fractions decompositions that arise and put up with complex coefficients.

16. **Example** ([1, Example 5.3.3]: Distinct roots)**.** Solve the initial-value problem

$$x' - x = 2\sin t \qquad x(0) = 0.$$

Note first that the method of undetermined coefficients tells us to expect no terms other than e^t, $\cos t$ and $\sin t$ in the solution (or: no terms other than e^t, e^{it} and e^{-it}). This was the purpose of Remark 15.

Applying $\mathscr{L}$ we get

$$(s-1)\mathscr{L}[x] = 2\mathscr{L}\Big[\frac{i}{2}(e^{-it}-e^{it})\Big] = \frac{i}{s+i} - \frac{i}{s-i} = \frac{2}{(s-i)(s+i)}$$

and hence

$$x = \mathscr{L}^{-1}\left[\frac{2}{(s-1)(s-i)(s+i)}\right].$$

To find x we look for the partial-fractions decomposition of $\mathscr{L}[x]$:

$$\frac{2}{(s-1)(s+i)(s-i)} = \frac{A}{s-1} + \frac{B}{s-i} + \frac{C}{s+i}$$

To determine A, B, and C, clear fractions and then insert the values of s for which the original denominator is zero, that is, $s = 1, \pm i$.

$$2 = A(s+i)(s-i) + B(s-1)(s+i) + C(s-1)(s-i)$$

For $s = 1$ this gives $2 = A(1+i)(1-i) = A(1-(i^2)) = 2A$, so $A = 1$.

For $s = i$ this gives

$$2 = B(i-1)(i+i) = 2B(i-1)i = 2B(i^2-i) = 2B(-1-i),$$

so $B = \dfrac{1}{-1-i} = -\dfrac{1}{2} + \dfrac{i}{2}$.

Finally, for $s = -i$ we get

$$2 = C(-i-1)(-i-i) = 2C(-i-1)(-i) = 2C((-i)^2+i) = 2C(-1+i),$$

so $C = \dfrac{1}{-1+i} = \dfrac{1}{-1+i}\dfrac{-1-i}{-1-i} = \dfrac{-1-i}{(-1)^2-i^2} = -\dfrac{1}{2} - \dfrac{i}{2}$.

17. **Remark.** Time-saver: B and C are complex conjugates. This is no accident!

We thus obtain the partial-fractions decomposition

$$\frac{2}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{-\frac{1}{2}+\frac{i}{2}}{s-i} + \frac{-\frac{1}{2}-\frac{i}{2}}{s+i}$$

Note that the two summands involving i are complex conjugates of each other, so they sum to a real number. This is why we *expect* their numerators B and C to be complex conjugates.

Now we "untransform" and sort terms:

$$\begin{aligned}
x &= \mathscr{L}^{-1}\left[\frac{1}{s-1} + \frac{-\frac{1}{2}+\frac{i}{2}}{s-i} + \frac{-\frac{1}{2}-\frac{i}{2}}{s+i}\right] \\
&= e^t + \big(-\frac{1}{2}+\frac{i}{2}\big)e^{it} + \big(-\frac{1}{2}-\frac{i}{2}\big)e^{-it} \\
&= e^t - \frac{1}{2}(e^{it}+e^{-it}) + \frac{i}{2}(e^{it}-e^{-it}) \\
&= e^t - \cos t - \sin t
\end{aligned}$$

We conclude that the initial-value problem

$$Dx - x = 2\sin t \qquad x(0) = 0.$$

has the unique solution

$$x = e^t - \cos t - \sin t.$$

The form matches with our expectations from the method of undetermined coefficients (Remark 15), so one only needs to check the initial condition to verify that this is correct: $x(0) = e^0 - \cos 0 - \sin 0 = 1 - 1$, as required.

18. **Example** ([1, Example 5.6.3]: Pair of double complex roots)**.**
Find $\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$.

We rewrite

$$\frac{s}{(s^2+1)^2} = \frac{s}{(s-i)^2(s+i)^2}$$

and produce the partial-fractions decomposition:

$$\frac{s}{(s-i)^2(s+i)^2} = \frac{A}{s-i} + \frac{B}{(s-i)^2} + \frac{C}{s+i} + \frac{D}{(s+i)^2}$$

becomes

$$s = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2.$$

Set $s = i$ to get

$$i = B(2i)^2 = -4B, \text{ hence } B = -\frac{i}{4}.$$

Set $s = -i$ to get

$$-i = D(-2i)^2 = -4D, \text{ hence } D = \frac{i}{4}.$$

Before going on we consolidate the corresponding terms:

$$\begin{aligned} B(s+i)^2 + D(s-i)^2 &= -\frac{i}{4}(s+i)^2 + \frac{i}{4}(s-i)^2 \\ &= \frac{i}{4}\big(-(s^2+2si-1) + (s^2-2si-1)\big) = -\frac{i}{4}(4si) = s. \end{aligned}$$

Therefore we can rewrite

$$s = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2$$

as

$$0 = A(s-i)(s+i)^2 + C(s+i)(s-i)^2.$$

This means that we can (and hence must!) take $A = C = 0$.

Therefore,

$$\mathscr{L}^{-1}\Big[\frac{s}{(s-i)^2(s+i)^2}\Big] = \mathscr{L}^{-1}\Big[-\frac{i/4}{(s-i)^2}+\frac{i/4}{(s+i)^2}\Big]$$
$$= -\frac{i}{4}te^{it}+\frac{i}{4}te^{-it} = \frac{t}{2}\Big[\frac{e^{it}-e^{-it}}{2i}\Big] = \frac{t}{2}\sin t.$$

19. **Example** ([1, Example 5.6.5]: Application to initial-value problem)**.** Solve

$$(D^2+1)x = \cos t, \qquad x(0) = x'(0) = 0.$$

Apply $\mathscr{L}$ to both sides to get

$$\underbrace{(s^2+1)}_{=(s-i)(s+i)}\mathscr{L}[x] = \frac{1}{2}\mathscr{L}[e^{it}+e^{-it}] = \frac{1}{2}\Big(\frac{1}{s-i}+\frac{1}{s+i}\Big) = \frac{s}{(s-i)(s+i)}$$

Thus, $x = \mathscr{L}^{-1}\Big[\frac{s}{(s-i)^2(s+i)^2}\Big] = \frac{t}{2}\sin t$ using the previous example.

20. **Example** ([1, Example 5.6.4]: Same pair of double complex roots)**.** Find $\mathscr{L}^{-1}\Big[\frac{1}{(s^2+1)^2}\Big]$.

We write $\frac{1}{(s^2+1)^2} = \frac{1}{(s-i)^2(s+i)^2}$ and produce the partial-fractions decomposition:

$$\frac{1}{(s-i)^2(s+i)^2} = \frac{A}{s-i}+\frac{B}{(s-i)^2}+\frac{C}{s+i}+\frac{D}{(s+i)^2}$$

becomes

$$1 = A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2.$$

This can be solved for A, B, C and D by multiplying out the right-hand side, sorting by powers of s and comparing coefficients to get 4 linear equations in these 4 unknowns. We try sampling useful s-values first to break the problem up.

Set $s = i$ to get

$$1 = B(2i)^2 = -4B, \text{ hence } B = -\frac{1}{4}.$$

Set $s = -i$ to get

$$1 = D(-2i)^2 = -4D, \text{ hence } D = -\frac{1}{4}.$$

Consolidate:

$$B(s+i)^2 + D(s-i)^2 = -\frac{1}{4}\big((s^2+2si-1)+(s^2-2si-1)\big) = -\frac{1}{2}(s^2-1).$$

Therefore,

$$\begin{aligned}1 &= A(s-i)(s+i)^2 + B(s+i)^2 + C(s+i)(s-i)^2 + D(s-i)^2\\ &= A(s-i)(s+i)^2 + C(s+i)(s-i)^2 - \frac{1}{2}(s^2-1)\end{aligned}$$

and, adding $\frac{1}{2}(s^2-1)$ to both sides,

$$\begin{aligned}\frac{1}{2}(s^2+1) = 1 + \frac{1}{2}(s^2-1) &= A(s-i)(s+i)^2 + C(s+i)(s-i)^2\\ &= A(s^2+1)(s+i) + C(s^2+1)(s-i).\end{aligned}$$

Dividing by the common factor s^2+1, this becomes

$$\frac{1}{2} = A(s+i) + C(s-i) = \underbrace{(A+C)s + i(A-C)}_{\text{sorted by powers of } s}.$$

Comparing coefficients of like powers of s we find that

$$A + C = 0 \quad \text{and} \quad \frac{1}{2} = i(A-C) = 2iA,$$

so

$$A = \frac{1}{4i} \quad \text{and} \quad C = -\frac{1}{4i}.$$

Inserting these into the partial-fractions decomposition we find

$$\begin{aligned}\frac{1}{(s-i)^2(s+i)^2} &= \frac{A}{s-i} + \frac{B}{(s-i)^2} + \frac{C}{s+i} + \frac{D}{(s+i)^2}\\ &= \frac{1/4i}{s-i} - \frac{1/4}{(s-i)^2} - \frac{1/4i}{s+i} - \frac{1/4}{(s+i)^2}\end{aligned}$$

This gives us

$$\begin{aligned}\mathscr{L}^{-1}\Big[\frac{1}{(s^2+1)^2}\Big] &= \mathscr{L}^{-1}\Big[\frac{1/4i}{s-i} - \frac{1/4}{(s-i)^2} - \frac{1/4i}{s+i} - \frac{1/4}{(s+i)^2}\Big]\\ &= \frac{1}{4i}e^{it} - \frac{t}{4}e^{it} - \frac{1}{4i}e^{-it} - \frac{t}{4}e^{-it}\\ &= \frac{1}{2}\frac{e^{it}-e^{-it}}{2i} - \frac{t}{2}\frac{e^{it}+e^{-it}}{2}\\ &= \frac{1}{2}\sin t - \frac{t}{2}\cos t.\end{aligned}$$

21. **Example** (Initial-value problem producing triple complex roots)**.** Solve

$$(D^2+9)^2 x = -4\sin 3t, \qquad x(0) = x'(0) = x''(0) = 0,\ x'''(0) = 1.$$

We first note that we should expect only the functions $\sin 3t$, $\cos 3t$, $t\sin 3t$, $t\cos 3t$, $t^2\sin 3t$, $t^2\cos 3t$ in the ultimate solution (Remark 15).

Apply $\mathscr{L}$ to get

$$(s^2+9)^2\mathscr{L}[x]-1=-4\frac{3}{s^2+9},$$

so

$$\mathscr{L}[x]=\frac{1}{(s^2+9)^2}-\frac{12}{(s^2+9)^3}=\frac{\overbrace{s^2-3}^{=(s^2+9)-12}}{\underbrace{(s^2+9)}_{=(s-3i)(s+3i)}{}^3} \tag{2}$$

We need to decompose the right-hand side as follows:

$$\frac{s^2-3}{(s-3i)^3(s+3i)^3}=\frac{A}{s-3i}+\frac{B}{s+3i}+\frac{C}{(s-3i)^2}+\frac{D}{(s+3i)^2}+\frac{E}{(s-3i)^3}+\frac{F}{(s+3i)^3}.$$

Clear fractions to get

$$\begin{aligned}s^2-3=&A(s-3i)^2(s+3i)^3+B(s-3i)^3(s+3i)^2\\&+C(s-3i)(s+3i)^3+D(s-3i)^3(s+3i)\\&+E(s+3i)^3+F(s-3i)^2.\end{aligned}$$

For $s=3i$ this becomes

$$E=\frac{-9-3}{(6i)^3}=\frac{2\cdot 6}{6\cdot 3\cdot 2\cdot 6i}=\frac{1}{18i},$$

and F must be the complex conjugate: $F=-\frac{1}{18i}$. We consolidate these 2 terms first:

$$\frac{E}{(s-3i)^3}+\frac{F}{(s+3i)^3}=\frac{1/18i}{(s-3i)^3}-\frac{1/18i}{(s+3i)^3}=\frac{\overbrace{s^2-3}^{=\frac{(s+3i)^3}{18i}-\frac{(s-3i)^3}{18i}}}{(s^2+9)^3}.$$

Note from equation (2) above that we have just stumbled upon the partial-fractions decomposition we need, that is, that (using equation (1)) we have

$$x=\mathscr{L}^{-1}\left[\frac{s^2-3}{(s^2+9)^3}\right]=\frac{1}{18i}\mathscr{L}^{-1}\left[\frac{1}{(s-3i)^3}-\frac{1}{(s+3i)^3}\right]=\frac{1}{18}\frac{t^2e^{3it}-t^2e^{-3it}}{2i}.$$

In short, the solution to the given initial-value problem is

$$x=\frac{1}{18}t^2\sin 3t.$$

By inspection (really!) one can see that this satisfies the initial condition.

This last example illustrates that we can in this way handle complex roots of any multiplicity, not just double complex roots.

REFERENCES

[1] Martin M. Guterman, Zbigniew H. Nitecki, *Differential Equations – A First Course*, 3^{rd} ed., Saunders (1992).