

## PHASE PORTRAITS OF LINEAR SYSTEMS

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For our purposes phase portraits of linear second-order systems are of interest primarily when we might use the Hartman–Grobman Theorem [1, page 353]. This means that we can restrict attention to those cases where no eigenvalue has zero real part.

### 1. REAL EIGENVALUES

If none of the two eigenvalues is zero then there are 3 cases: Both eigenvalues are negative, both are positive, or there is one each.

1.1. **One positive and one negative eigenvalue.** This is described in Example 4.2.1 of [1]. Here the origin is a saddle: unstable but neither a repeller nor an attractor.

Draw the two *eigenlines* (the lines defined by eigenvectors). Put two “outward” arrows on the eigenline that corresponds to the positive eigenvalue and two “inward” arrows on the eigenline for the negative eigenvalue. Fill in the “quadrants” between eigenlines with “hyperbolic” curves with arrows that are consistent with the ones on the eigenlines.

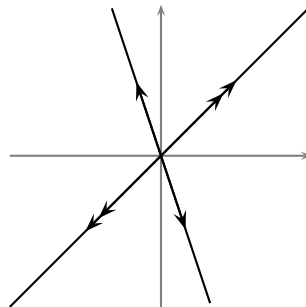
See Figures 4.8 and 4.9 in [1].

1.2. **Two positive eigenvalues.** There are two possibilities here: The eigenvalues are distinct, so we really have two positive eigenvalues, or they are the same, so there is really only one eigenvalue, and it is positive.

In either case the origin is a repeller, and hence unstable.

1.2.1. *Two distinct positive eigenvalues.* This is described in Example 4.2.2 of [1].

Draw both eigenlines. The eigenline corresponding to the larger of the two eigenvalues is the “fast” eigenline; we put *double* outward arrows on each half of this line to remind us that this is the fast one. The other eigenline is “slow” and gets single outward arrows. In Example 4.2.1 of [1] this looks as follows:



Now draw “parabolic” curves as follows: They are tangent to the slow eigenline at the origin, and they have the slope of the fast eigenline far from the origin. Provide them all with *outward* arrows to get Figure 4.11 in [1].

Note that if the two eigenlines make a small angle, then the parabolic curves in the thin wedge bend very little. That makes them a tad hard to draw. The other curves make a big turn. This determines the picture.

1.2.2. *A single positive eigenvalue with one eigenvector.* This is easiest to draw if one thinks of the two eigenlines from the previous case having merged into a single one. Then two thin wedges have altogether disappeared, and the remaining curves are halves of parabolic ones with *outward* arrows. They must be tangent to the single eigenline at the origin and have the same slope as the single eigenline far from it. In doing so they make a half-turn.

Since we don't know how the fast eigenline merged into the slow one, there are two possibilities: "clockwise" and "counterclockwise" turns. To see which is the case, analyze the behavior at a test point that is *not* on the eigenline. It is convenient to choose it on a coordinate axis.

**Example 1.** Consider the system  $D\vec{x} = A\vec{x}$ , where  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . The only eigenvalue is 1, and the corresponding eigenline is the horizontal coordinate axis. Do the "loose" ends of the half-parabolas point left or right? Take the point  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  on the vertical axis. At this point the direction of motion is given by  $D\vec{x}$ , and we have

$$D\vec{x} = A\vec{x} = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore the motion is right (and up, but that's less important), so the half-parabolas on top open to the right, and those at the bottom (by symmetry) open to the left.

1.2.3. *A single positive eigenvalue with two eigenvectors.* This is easy; see Figure 4.12 in [1] and the prose right before it. Draw radial lines with outward arrows. Make sure they look like lines and not curved.

1.3. **Two negative eigenvalues.** This works much like for two positive eigenvalues, but now all arrows point inward - the origin is an attractor and hence also stable.

1.3.1. *Two distinct negative eigenvalues.* Draw the eigenlines with inward-pointing arrows. The "fast" eigenline gets double arrows. This is the eigenline that corresponds to the eigenvalue furthest from 0, that is, the eigenvalue with largest absolute value.

Once the slow-fast information is in the picture, the method is as before: Draw "parabolic" curves that are tangent to the slow eigenline at the origin and have the slope of the fast eigenline far from the origin. Provide them all with *inward* arrows.

1.3.2. *A single negative eigenvalue with one eigenvector.* As for the case of a single positive eigenvalue with one eigenvector, we get one eigenline and curves that are halves of parabolic ones with outward arrows. They are tangent to the single eigenline at the origin and have the same slope as the single eigenline far from it. In doing so they make a half-turn. Whether these curves bend one way or the other is decided with a test point as in Example 1. Keep in mind that the arrows are *inward*. See Figure 4.13 in [1].

1.3.3. *A single negative eigenvalue with two eigenvectors.* Draw radial lines with inward arrows. See Figure 4.12 in [1].

## 2. COMPLEX EIGENVALUES (WITH NONZERO IMAGINARY PART)

These come in complex-conjugate pairs, so both eigenvalues have the same real part. Since we are interested in those cases with nonzero real part, there are two possibilities: positive or negative real part.

**2.1. Positive real part.** The origin is a repeller, hence unstable.

The solution curves are spirals with *outward* arrows. To determine whether the spirals turn clockwise or counterclockwise, use test points as in Example 1. If the test point is on the horizontal axis one needs to know whether the motion there goes up or down; if the test point is on the vertical axis one needs to check for left or right motion.

We do not care much about other aspects of the shape of these spirals.

**2.2. Negative real part.** The origin is an attractor, hence stable.

Same picture as before, with *inward* arrows. A test point will distinguish between clockwise and counterclockwise motion. See Figure 4.16 in [1].

## 3. EIGENVALUES WITH ZERO REAL PART

Whenever there is an eigenvalue with zero real part, the Hartman–Grobman Theorem cannot be applied. Therefore these cases are not of great interest in our applications. However, the case of imaginary eigenvalues is of interest for a different reason, and because [1] does not cover all cases of zero eigenvalues, we mention some of these for completeness.

### 3.1. Imaginary eigenvalues.

**3.1.1. Introduction.** The case of imaginary eigenvalues is not of interest when trying to use the Hartman–Grobman theorem, but it is of interest for a separate reason. The leading cases discussed in this note (those with two distinct eigenvalues) have in common that their qualitative behavior does not change when perturbing the matrix. Put differently, a sufficiently small change to a matrix in any one of these cases will not move you out of that particular case.

It is not necessarily the case that the qualitative behavior of the solution is insensitive to perturbations in the matrix when the eigenvalues are imaginary. However, if the system is area-preserving (an important concept in linear algebra that is outside the scope of this course, but which simply means that the sum of the eigenvalues of the matrix is zero), then small changes to the matrix do not have a drastic effect on the qualitative behavior of the solution. This in turn is of interest because this is the case for systems that arise in Hamiltonian mechanics.

**3.1.2. Method.** Drawing good enough phase portraits for linear second-order systems with imaginary eigenvalues is easy: Draw closed curves around the origin (it is not particularly important exactly what they look like, provided they are symmetric around the origin) and add arrows in a direction suggested by a test point on an axis. See Figures 4.17(a) and (b) in [1].

**3.1.3. Digression.** *Although this is not needed*, there is an easy way to draw a more accurate picture: If  $\vec{v} + i\vec{w}$  is an eigenvector (with  $\vec{v}$  and  $\vec{w}$  being real), then there is a solution through the two points  $\vec{v}$  and  $\vec{w}$ . Since solutions move on ellipses centered at the origin, knowing these two points determines one ellipse in the phase portrait, and all other solution curves are scaled copies of it.

To see why this is so, note that if  $\pm\beta i$  are the eigenvalues and  $\vec{v} \pm i\vec{w}$  are corresponding eigenvectors, then our usual procedure gives the corresponding solutions

$$\vec{x}_{\pm}(t) = (\cos \beta t \pm i \sin \beta t)(\vec{v} \pm i\vec{w}) = (\cos \beta t)(\vec{v} \pm i\vec{w}) + (\sin \beta t)(-\vec{w} \pm i\vec{v}).$$

Then  $\vec{x}(t) = \frac{\vec{x}_+(t) - \vec{x}_-(t)}{2i} = \cos \beta t \vec{w} + \sin \beta t \vec{v}$  is also a solution, and it satisfies  $\vec{x}(0) = \vec{w}$ ,  $\vec{x}(\pi/(2\beta)) = \vec{v}$ .

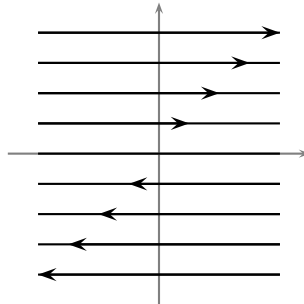
To digress further, we remark that the same reasoning in case of eigenvalues  $\alpha \pm i\beta$  gives a solution through  $\vec{w}$  and  $e^{(\pi/2)(\alpha/\beta)} \vec{v}$ , which can help to understand the distortion of the spirals obtained in Section 2 above.

**3.2. Zero eigenvalues.** If one eigenvalue has zero real part, then there are two cases: If the eigenvalues are complex, then they form a conjugate pair and must hence both be imaginary. We just finished this case. If the eigenvalues are real, then having zero real part means being zero. The case where one eigenvalue is zero and the other one is not is treated in [1], and we now discuss the only remaining case: Both roots of the characteristic polynomial are zero. This is neither interesting nor important, but it is not included in [1].

**3.2.1. Two linearly independent eigenvectors.** In this case the matrix is the zero matrix, so  $D\vec{x} = \vec{0}$ , i.e., nothing moves, and every solution curve is a point.

**3.2.2. Only one linearly independent eigenvector.** The line determined by the eigenvector consists of fixed points, and all other solution curves are parallel to this line, moving in opposite directions on either side of the line.

For example, the general solution of  $D\vec{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{x}$  is generated by  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} t \\ 1 \end{pmatrix}$ .



#### REFERENCES

- [1] Martin M. Guterman, Zbigniew H. Nitecki, *Differential Equations – A First Course*, 3<sup>rd</sup> ed., Saunders (1992).