Math 38 Exam 1

22 February 2007

1. (a) $\frac{dx}{dt} = .05x$

(b) $x = x_0 e^{.05t}$ $\quad x(0) = \$1000 \Rightarrow x_0 = \$1000$

$x = 1000e^{.05t}$

$x(20) = 1000e^{(.05)(20)} \approx \boxed{\$2780}$

2. $x^2 \frac{dx}{dt} = t^2 \qquad x(1) = 2$

$x^2 dx = t^2 dt$

$x^3 = t^3 + C$

$x = \sqrt[3]{t^3 + C}$

$\sqrt[3]{1 + C} = 2$

$1 + C = 8$

$C = 7$

$\boxed{x = \sqrt[3]{t^3 + 7}}$

3. $\frac{dx}{dt} - tx = t \qquad x(0) = \frac{1}{2}$

$\frac{dx}{dt} = t(x+1)$

$\frac{dx}{x+1} = t\,dt$

$\ln|x+1| = \frac{1}{2}t^2 + C$

$x + 1 = ke^{\frac{1}{2}t^2}$

$\frac{1}{2} + 1 = k$

$\boxed{x = -1 + \frac{3}{2}e^{\frac{1}{2}t^2}}$

4. (c) The theorem does not apply at $t=0$ because

$$f(t,x) = \frac{2x}{t}$$

and

$$f_x(t,x) = \frac{2}{t}$$

are not defined at $t=0$.

4. (d)

At $t=0$,

$$tx' - 2x = 0$$

$$\Rightarrow \quad 0\,x'(0) - 2x(0) = 0$$

$$\Rightarrow \quad x(0) = 0$$

4. (a) $x = ct^2$ for $t \neq 0$

$x' = 2ct$

$tx' - 2x = 2ct^2 - 2ct^2 = 0$

for any t including $t < 0$ and $t > 0$.

(b) The piece-wise function

$$x(t) = \begin{cases} c_1 t^2, & t \geq 0 \\ c_2 t^2, & t < 0 \end{cases}$$

satisfies

$$tx' - 2x = 0 \quad \text{on } (-\infty, \infty)$$

even when $c_1 \neq c_2$ because from part (a), for any constant c, $x = ct^2$ is a solution for $t < 0$ and for $t \geq 0$ and because

$$x'(t) = \begin{cases} 2c_1 t, & t \geq 0 \\ 2c_2 t, & t < 0 \end{cases}$$

is well defined (and continuous) at $t = 0$.

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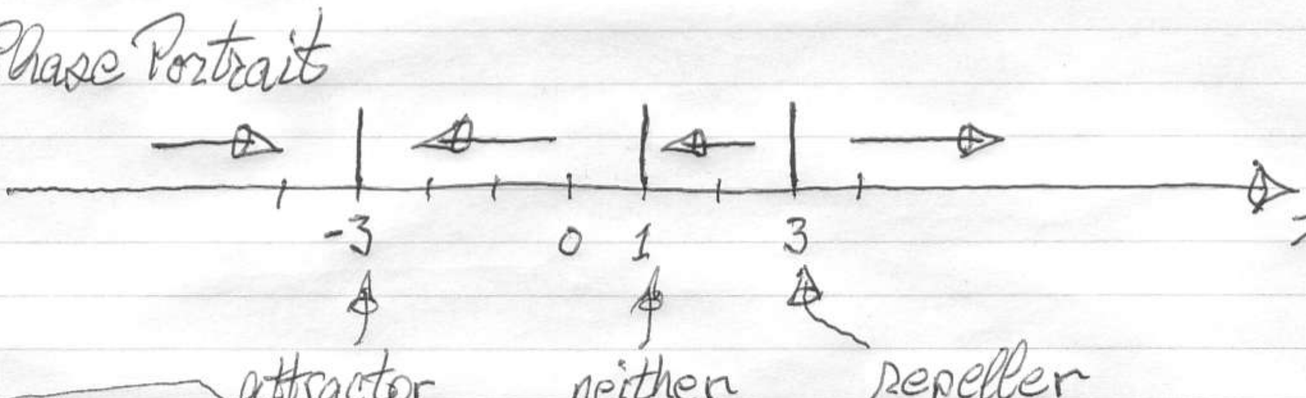
5. $$\frac{dx}{dt} = (x^2-9)(1-x)^2$$

Solve
$$(x^2-9)(1-x)^2 = 0$$
to get the equilibrium solutions

$$\boxed{x = \pm 3, \quad x = 1}$$

	$x+3$	$(1-x)^2$	$x-3$	$\frac{dx}{dt}$
$x<-3$	$-$	$+$	$-$	$+$
$-3<x<1$	$+$	$+$	$-$	$-$
$1<x<3$	$+$	$+$	$-$	$-$
$x>3$	$+$	$+$	$+$	$+$

Phase Portrait



Equilibria:

$x=-3$ stable, attractor,

$x=1$ unstable, neither repeller nor attractor.

$x=3$ unstable, repeller.

6. $h_1(t) = t^2$, $h_2(t) = t^{-1}$, $h_3(t) = 1$

(a)
$$W[h_1, h_2, h_3](t) = \begin{vmatrix} t^2 & t^{-1} & 1 \\ 2t & -t^{-2} & 0 \\ 2 & 2t^{-3} & 0 \end{vmatrix}$$

$$W[h_1, h_2, h_3](1) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} = 6 \neq 0$$

$W[h_1, h_2, h_3](1) \neq 0$

$\Rightarrow$ linear independence

$\Rightarrow$ h_1, h_2, h_3 form the general solutn

$$\boxed{H(t) = c_1 t^2 + c_2 t^{-1} + c_3}$$

6. (b) Find a particular solution of the form $p(t) = kt^3$

$p'(t) = 3kt^2$, $p''(t) = 6kt$, $p'''(t) = 6k$

$$(t^2 D^3 + 2tD^2 - 2D)\,p(t) = 6kt^2 + 12kt^2 - 6kt^2 = t^2$$

$$12k = 1 \Rightarrow k = \frac{1}{12}$$

General solutn:

$$\boxed{x(t) = p(t) + H(t) = \frac{1}{12}t^3 + c_1 t^2 + c_2 t^{-1} + c_3}$$

Alternative solution for 6(b)

6.(b) $\left(t^2 D^3 + 2t D^2 - 2D\right) x = t^2$

(N) $\left(D^3 + \frac{2}{t} D^2 - \frac{2}{t^2} D\right) x = 1$

$$w_1' t^2 + w_2' t^{-1} + w_3' = 0$$

$$\left.\begin{aligned} 2 w_1' t - w_2' t^{-2} &= 0 \\ 2 w_1' + 2 w_2' t^{-3} &= 1 \end{aligned}\right\} \quad 3 w_2' t^{-2} = t \qquad \boxed{w_2' = \frac{1}{3} t^3}$$

$$\begin{bmatrix} t^2 & t^{-1} & 1 \\ 2t & -t^{-2} & 0 \\ 2 & 2t^{-3} & 0 \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \\ w_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w_1' = \frac{1}{2} - w_2' t^{-3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \qquad \boxed{w_1' = \frac{1}{6}}$$

$$w_3' = -w_1' t^2 - w_2' t^{-1} = -\frac{1}{6} t^2 - \frac{1}{3} t^2 \qquad \boxed{w_3' = -\frac{1}{2} t^2}$$

$$\boxed{w_1(t) = \frac{1}{6} t \qquad w_2(t) = \frac{1}{12} t^4 \qquad w_3(t) = -\frac{1}{6} t^3}$$

$$p(t) = w_1(t) h_1(t) + w_2(t) h_2(t) + w_3(t) h_3(t)$$

$$= \frac{1}{6} t^3 + \frac{1}{12} t^3 - \frac{1}{6} t^3 = \frac{1}{12} t^3$$

$$x(t) = p(t) + H(t)$$

$$\boxed{x(t) = \frac{1}{12} t^3 + c_1 t^2 + \frac{c_2}{t} + c_3}$$

#7 $(D-1)^2(D+3)x = 0$ $\quad x(0)=0,\ x'(0)=1,\ x''(0)=0$

has genrl solutn

$$x(t) = c_1 e^{t} + c_2 t e^{t} + c_3 e^{-3t}$$
$$x'(t) = c_1 e^{t} + c_2(t+1)e^{t} - 3c_3 e^{-3t}$$
$$x''(t) = c_1 e^{t} + c_2(t+2)e^{t} + 9c_3 e^{-3t}$$

$c_1 + c_3 = 0 \Rightarrow c_3 = -c_1$

$\left.\begin{array}{l} c_1 + c_2 - 3c_3 = 1 \\ c_1 + 2c_2 + 9c_3 = 0 \end{array}\right\} \quad c_1 - 15c_3 = 2$

$16c_1 = 2$, $\quad c_1 = \frac{1}{8}$, $\quad c_3 = -\frac{1}{8}$

$c_2 = 1 + 3c_3 - c_1 = 1 - \frac{3}{8} - \frac{1}{8} \qquad c_2 = \frac{1}{2}$

$$\boxed{x(t) = \frac{1}{8}e^{t} + \frac{1}{2}te^{t} - \frac{1}{8}e^{-3t}}$$

#7 Alternative solver for c_1, c_2, c_3.

Solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -3 \\ 1 & 2 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

By Cramer's Rule,

$$c_1 = \frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & 2 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -3 \\ 1 & 2 & 9 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}}{\underbrace{\begin{vmatrix} 1 & -3 \\ 2 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}_{(9+6)+(2-1)}} = \frac{2}{16} = \frac{1}{8}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -3 \\ 1 & 0 & 9 \end{vmatrix}}{16} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}}{16} = \frac{9-1}{16} = \frac{1}{2}$$

$$c_3 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}}{16} = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{16} = \frac{-2}{16} = -\frac{1}{8}$$

2/20/2007

8. (a) $(9D^2-1)x = t$

(H) $(9D^2-1)x = 0$

$$H(t) = c_1 e^{\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t}$$

(H*) $A(D)\,P(D) = 0$

where $A(r)$ has root 0 with multply 2

$$H^*(t) = c_1 e^{\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t} + c_3 + c_4 t$$

(N) $p(t) = k_1 + k_2 t$

$$p'(t) = k_2, \quad p''(t) = 0$$

$$(9D^2-1)p(t) = -k_1 - k_2 t \overset{set}{=} t \Rightarrow k_1 = 0, \; k_2 = -1$$

$$p(t) = -t$$

$$\boxed{x(t) = -t + c_1 e^{\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t}}$$

Check: $x'(t) = -1 + \frac{1}{3} c_1 e^{\frac{1}{3}t} - \frac{1}{3} c_2 e^{-\frac{1}{3}t}$

$x''(t) = \frac{1}{9} c_1 e^{\frac{1}{3}t} + \frac{1}{9} c_2 e^{-\frac{1}{3}t}$

$$(9D^2-1)x = c_1 e^{\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t} + t - c_1 e^{\frac{1}{3}t} - c_2 e^{-\frac{1}{3}t} = t \checkmark$$

8(b) $(D^2+1)x = \sin t$

(H) $(D^2+1)x = 0$

$$\boxed{H(t) = c_1 \cos t + c_2 \sin t}$$

(A) $A(D)\,P(D)\,x = 0$

$A(r)$ has roots $r = \pm i$ with multiplicity 1

$$H^*(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

(N) $p(t) = k_1 t \cos t + k_2 t \sin t$

$$p'(t) = k_1 \cos t + k_2 \sin t - k_1 t \sin t + k_2 t \cos t$$

$$\begin{aligned} p''(t) &= -k_1 \sin t + k_2 \cos t \\ &\quad -k_1 \sin t + k_2 \cos t \\ &\quad -k_1 t \cos t - k_2 t \sin t \\ &= -2k_1 \sin t + 2k_2 \cos t \\ &\quad -k_1 t \cos t - k_2 t \sin t \end{aligned}$$

8(b) (continued)

$$D^2 p(t) + p(t) = -2k_1 \sin t + 2k_2 \cos t$$
$$- k_1 t \cos t - k_2 t \sin t$$
$$+ k_1 t \cos t + k_2 t \sin t$$
$$= -2k_1 \sin t + 2k_2 \cos t$$
$$\overset{set}{=} \sin t \Rightarrow k_2 = 0$$
$$k_1 = -\tfrac{1}{2}$$

$$\boxed{p(t) = -\tfrac{1}{2} t \cos t}$$

Check: $p'(t) = -\frac{1}{2}\cos t + \frac{1}{2} t \sin t$

$$p''(t) = \tfrac{1}{2}\sin t + \tfrac{1}{2}\sin t + \tfrac{1}{2} t \cos t$$
$$= \sin t + \tfrac{1}{2} t \cos t$$

$$p''(t) + p(t) = \sin t \checkmark$$

$$x(t) = p(t) + H(t)$$

$$\boxed{x(t) = -\tfrac{1}{2} t \cos t + C_1 \cos t + C_2 \sin t}$$