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Exam 2, Math 38 $\boxed{\text{Solutions}}$ Monday, March 12, 2007

Differential Equations

1. $\mathcal{L}[t](s) = \int_0^\infty e^{-st}\, t\, dt$ Let $u = t$, $dv = e^{-st}\, dt$. Then $du = dt$, $v = -\frac{1}{s} e^{-st}$

$$= -\frac{1}{s} t e^{-st}\Big|_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$\boxed{\mathcal{L}[t](s) = \frac{1}{s^2} \quad \text{for } s > 0}$$

2. $$f(t) = \begin{cases} 2 & \text{for } t < 1 \\ 0 & \text{for } 1 \le t < 2 \\ t & \text{for } t \ge 2 \end{cases}$$

$$f(t) = 2 + u_1(t)(-2) + u_2(t)\, t$$

$$\boxed{f(t) = 2 - 2u_1(t) + u_2(t)\, t}$$

3. $$e^t * e^{2t} = \int_0^t e^{t-u} e^{2u}\, du$$

$$= e^t \int_0^t e^u\, du$$

$$= e^t (e^t - 1)$$

$$\boxed{e^t * e^{2t} = e^{2t} - e^t}$$

4. (a) $\mathcal{L}\left[t e^{3t} \cos 2t\right](s) = -\frac{d}{ds} \mathcal{L}\left[e^{3t} \cos 2t\right](s)$

$$= -\frac{d}{ds} \mathcal{L}\left[\cos 2t\right](s-3)$$

$$= -\frac{d}{ds}\left(\frac{s-3}{(s-3)^2+4}\right) = -\frac{d}{ds}\left(\frac{s-3}{s^2-6s+13}\right)$$

$$\boxed{\mathcal{L}\left[t e^{3t} \cos 2t\right](s) = \frac{s^2-6s+5}{\left(s^2-6s+13\right)^2}}$$

4. (b)

$$f(t) = \begin{cases} t-3 & \text{for } t < 2 \\ 0 & \text{for } t \geq 2 \end{cases}$$

$$= \left(1 - u_2(t)\right)(t-3)$$

$$\mathcal{L}\left[f(t)\right](s) = \mathcal{L}\left[t-3\right](s) - \mathcal{L}\left[u_2(t)(t-3)\right](s)$$

$$= \frac{1}{s^2} - \frac{3}{s} - e^{-2s} \mathcal{L}\left[(t+2)-3\right](s)$$

$$\boxed{\mathcal{L}\left[f(t)\right](s) = \frac{1}{s^2} - \frac{3}{s} - e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right)}$$

5.(a) $$\frac{s+3}{s^2+2s+5} = \frac{s+3}{s^2+2s+1+4} = \frac{s+3}{(s+1)^2+4}$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{s^2+2s+5}\right](t) = \mathcal{L}^{-1}\left[\frac{s+3}{(s+1)^2+4}\right](t)$$

$$= e^{-t}\,\mathcal{L}^{-1}\left[\frac{s+2}{s^2+4}\right](t)$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s+3}{s^2+2s+5}\right](t) = e^{-t}\left(\cos 2t + \sin 2t\right)}$$

5(b) $$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) * \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t)$$

$$= 1 * \sin t = \int_0^t \sin u \, du$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right](t) = 1 - \cos t}$$

5(c) $$\mathcal{L}^{-1}\left[\frac{s+e^{-\pi s}}{s^2+1}\right](t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t) + u_\pi(t)\,\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t-\pi)$$

$$= \cos t + u_\pi(t)\sin(t-\pi)$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s+e^{-\pi s}}{s^2+1}\right](t) = \cos t - u_\pi(t)\sin t}$$

6. (a)

Solve $(D^2+4)x = 4$, $x(0)=0$, $x'(0)=5$

$$\mathcal{L}[D^2x+4x](s) = \mathcal{L}[4](s)$$

$$-5 + (s^2+4)\,\mathcal{L}[x](s) = \frac{4}{s}$$

$$\mathcal{L}[x](s) = \frac{4}{s(s^2+4)} + \frac{5}{s^2+4}$$

$$= \frac{5s+4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$A(s^2+4) + (Bs+C)s = 5s+4$

at $s=0$, $4A = 4 \Rightarrow A=1$

Coeff of s^2: $A+B=0 \Rightarrow B=-1$

Coeff of s ; $C=5$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s-5}{s^2+4}\right](t)$$

$$\boxed{x(t) = 1 - \cos 2t + \frac{5}{2}\sin 2t}$$

Check: $x(0) = 1-1=0$ ✓

$x'(t) = 2\sin 2t + 5\cos 2t$ $\quad x'(0)=5$ ✓

$x''(t) = 4\cos 2t - 10\sin 2t$

$x''+4x = (4-4)\cos 2t + \left(-10+\frac{20}{2}\right)\sin 2t + 4$ ✓

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~~#10 (a)~~ Solve $(D+1)x = \begin{cases} \sin t, & t < \pi \\ -\sin t, & t \geq \pi \end{cases}$

#6 (b)

$x(0) = 1$

$$= \left(1 - 2u_\pi(t)\right)\sin t$$

$$\mathcal{L}[Dx + x](s) = \mathcal{L}\left[\left(1 - 2u_\pi(t)\right)\sin t\right](s)$$

$$-x(0) + s\,\mathcal{L}[x](s) + \mathcal{L}[x](s)$$

$$= \mathcal{L}[\sin t](s) - 2\,\mathcal{L}[u_\pi(t)\sin t](s)$$

$$= \frac{1}{s^2+1} - 2e^{-\pi s}\,\mathcal{L}[\sin(t+\pi)](s)$$

$$= \frac{1}{s^2+1} + 2e^{-\pi s}\,\mathcal{L}[\sin t](s)$$

$$= \frac{1}{s^2+1} + \frac{2e^{-\pi s}}{s^2+1}$$

$$-1 + (s+1)\,\mathcal{L}[x](s) = \frac{1}{s^2+1} + \frac{2e^{-\pi s}}{s^2+1}$$

$$\mathcal{L}[x](s) = \frac{1}{(s+1)(s^2+1)} + \frac{2e^{-\pi s}}{(s+1)(s^2+1)} + \frac{1}{s+1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right](t) + 2u_\pi(t)\,\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right](t-\pi) + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)$$

~~#10 (a)~~ continued

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A(s^2+1) + (Bs+C)(s+1) = 1$$

Coefficient of s^2: $A + B = 0$ Let $s = -1$

s: $B + C = 0$ Then $2A = 1$

Constant term: $A + C = 1$ $\Rightarrow A = \frac{1}{2}$

$B = -\frac{1}{2}$

$C = \frac{1}{2}$

$$x(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s-1}{s^2+1}\right](t)$$

$$+ 2\,u_\pi(t)\left\{\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t-\pi) - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s-1}{s^2+1}\right](t-\pi)\right\}$$

$$+ \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)$$

$$= \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$+ \cancel{2}\,u_\pi(t)\left[e^{-(t-\pi)} - \cos(t-\pi) + \sin(t-\pi)\right]$$

$$= \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$+ u_\pi(t)\left[e^{-(t-\pi)} + \cos t - \sin t\right]$$

~~#10 (a)~~ continued

Check: $x(0) = \frac{3}{2} - \frac{1}{2} = 1$ ✓

$$x'(t) = -\frac{3}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\cos t$$
$$+ u_\pi(t)\left[-e^{-(t-\pi)} - \sin t - \cos t\right]$$

$$x'(t) + x(t) = \sin t - 2u_\pi(t)\sin t$$
$$= \left(1 - 2u_\pi(t)\right)\sin t$$